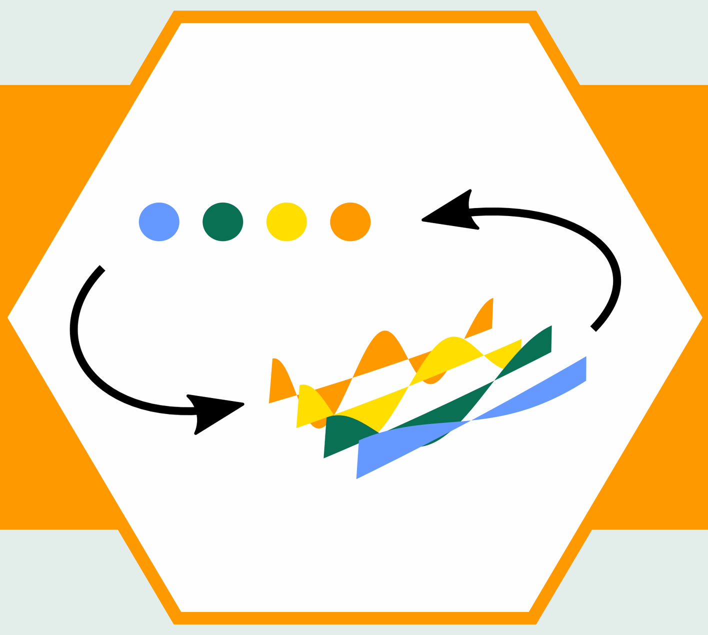

Many-Body Strategies for Multi-Qubit Gates

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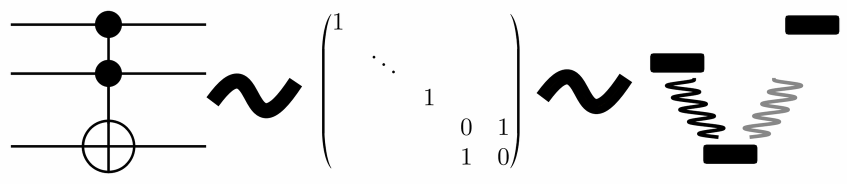
We observe that interesting multi-qubit gates, such as the Toffoli gate below, are very similar to what happens when we apply resonant driving in a many-body system. Both operations are highly entangling, but look clean and understandible as unitary matrix:



In order to accomplish such an operation on a linear chain of qubits, we require:
1) A many-body Hamiltonian which features a unique transition, and whose eigenstates extend over multiple qubits.
2) A driving field, which couples the transitioning states.
3) An *eigengate*, which maps between the computational basis and the eigenbasis of the many-body Hamiltonian.

This whole protocol can be summarized in the following 3 steps:

Based on
- **KG & KS**, *Many-body strategies for multi-qubit gates - quantum control through Krawtchouk chain dynamics*, ArXiv:**1707.05144**.
- **KG & KS**, *Manuscript in preparation.*

Resonant driving causes transitions only if the energy gap between states matches the driving frequency.

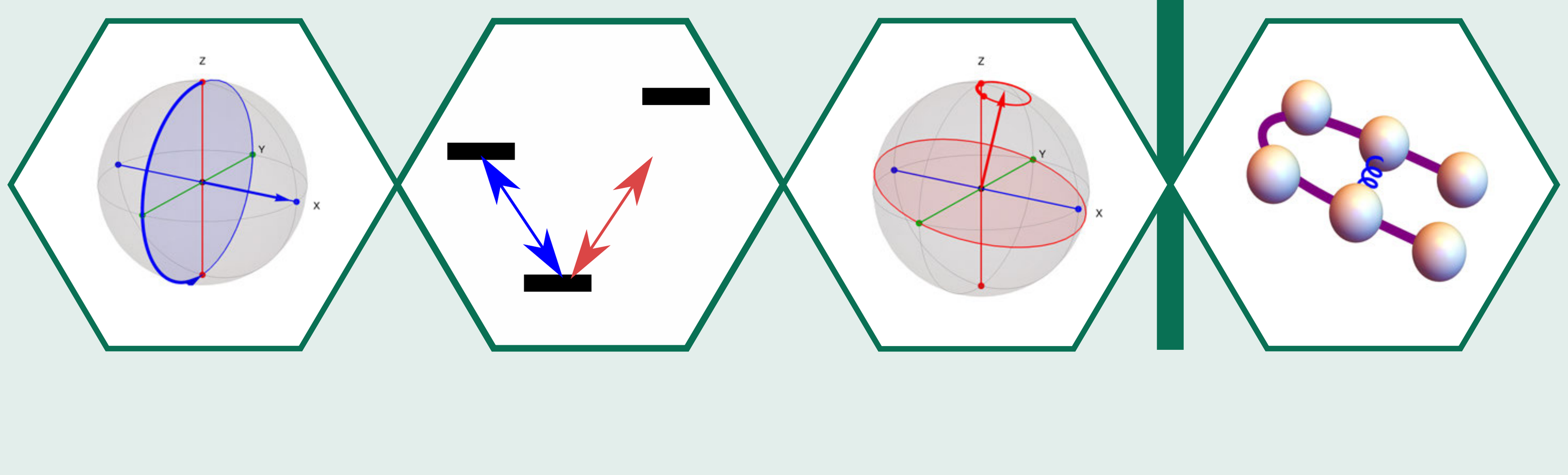
Assume we apply an oscillating field H_{drive} to a many-body background Hamiltonian H_{bg}:

$$H(t) = H_{\mathrm{bg}} + A' \cos(\omega t) H_{\mathrm{drive}}.$$

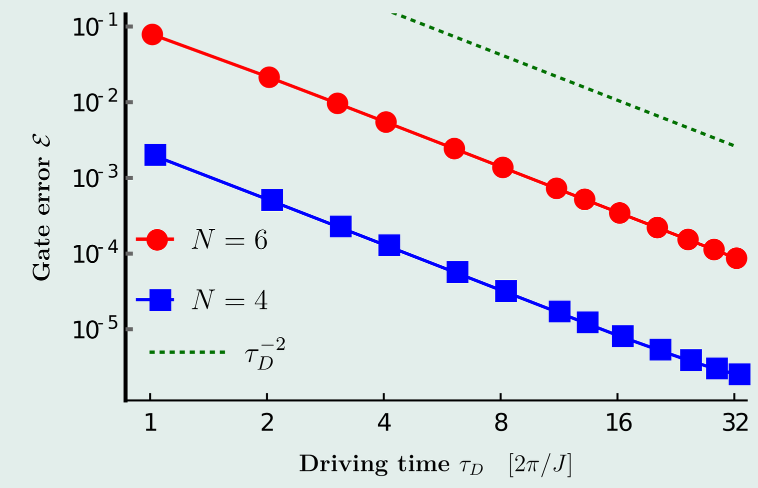
Looking at each pair of states as a two-level system, in a rotating frame at frequency ω, we may describe the evolution by

$$H_{\mathrm{TLS}} = \begin{pmatrix} \delta & A \\ A & -\delta \end{pmatrix}, \qquad \delta = E_1 - E_2 - \omega.$$

When the frequency ω is close to the energy gap (*on resonance*, $\delta = 0$) we find a perfect Pauli-X gate. In the opposite, *off-resonant* limit ($\delta >> A$), this merely applies a phase (Z)-gate and little amplitude is exchanged.



The error of the gate scales with total gate time as τ^{-2}.

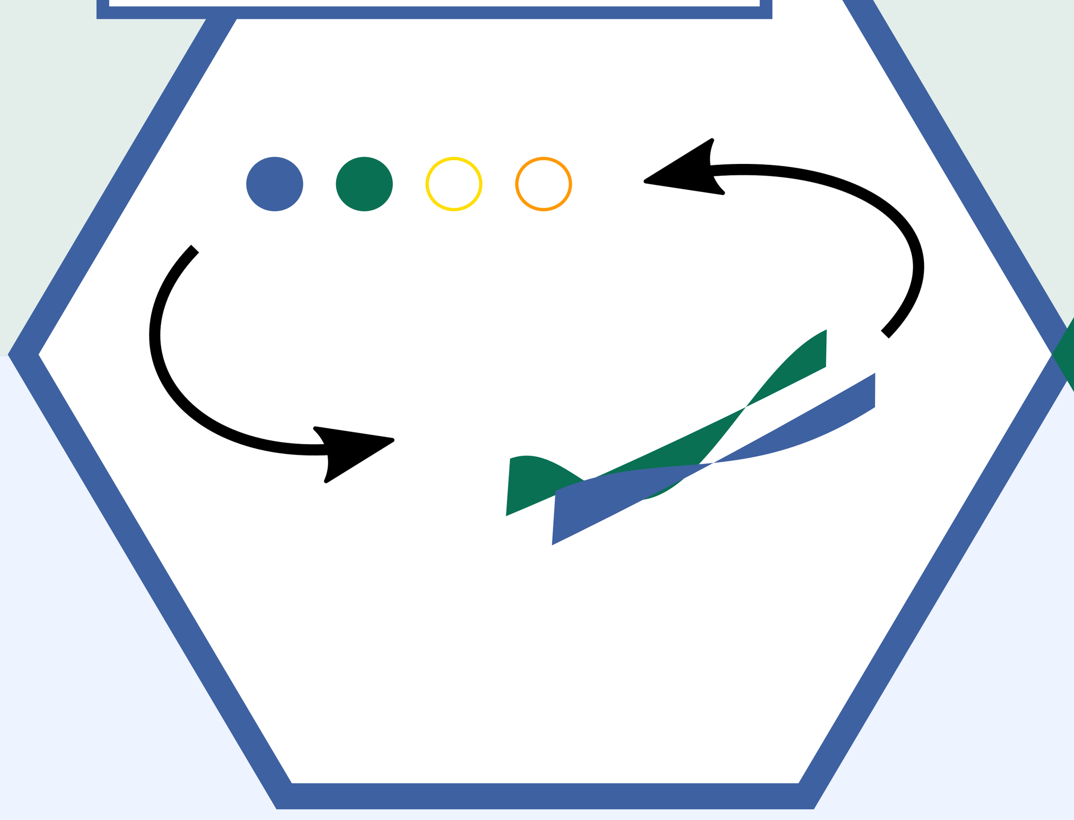


For driving in many-body systems, we cannot find exact solutions, but simulations allow us to predict fidelities of a driven gate. The off-resonant transitions, for which we assumed $\delta >> A$, become more precise if we decrease A, hence increase the time τ_D taken by the resonant transition. In general, we find that the error scales as $\mathcal{E} \propto \tau_D^{-2}$.
In the simulation above, we used the Krawtchouk Chain Hamiltonian H^x as background field, whilst driving with the field

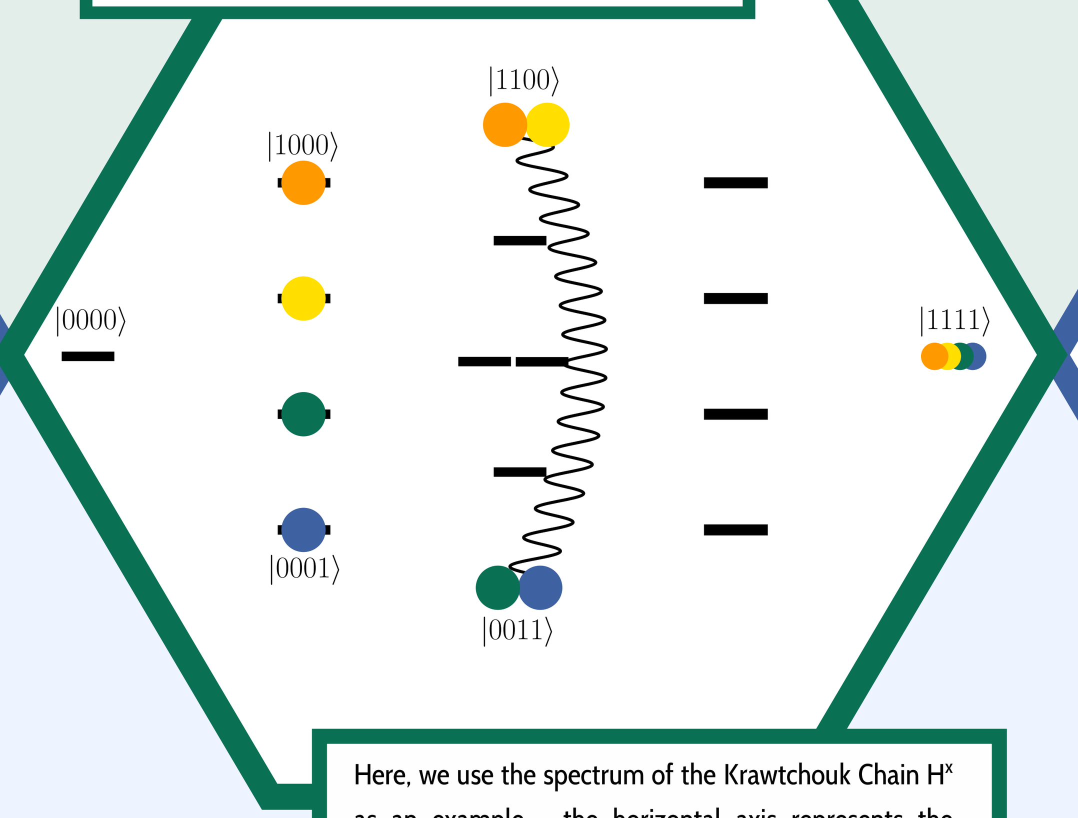
$$H_{\mathrm{drive}} = A\ \sigma_a^+ \sigma_b^- + h.c.$$

where (a, b) denotes qubits 1, 3 and 2, 4 when $N = 4$, or qubits 2, 5 when $N = 6$. A cartoon for the case $N = 6$ is displayed the on the left.

Step 1: We use an *eigengate* to make eigenstates

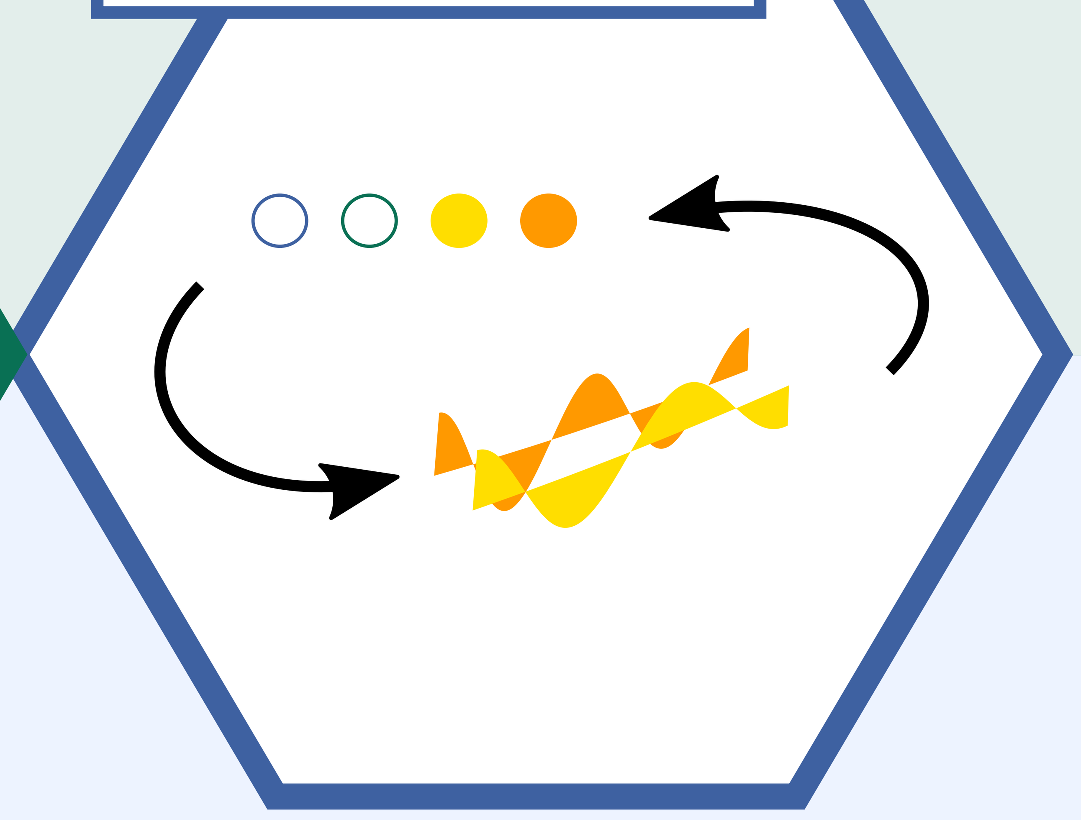


Step 2: Turn on the many-body Hamiltonian, and *resonantly drive* the unique transition.



Here, we use the spectrum of the Krawtchouk Chain H^x as an example - the horizontal axis represents the number of excitations.

Step 3: Another eigengate maps back to the computational basis



If $[H, A] = iB$ and $[H, B] = -iA$, then the Hamiltonian H can map eigenstates of A to eigenstates of B.

Our goal is to find an **eigengate** operation which maps eigenstates of operator A to eigenstates of operator B, where

- A is diagonal in the computational basis.
- B's eigenbasis consists of spatially extended, highly entangled states.

This mapping can be accomplished through a 'quench' (unitary time evolution) with some Hamiltonian H that satisfies:

$$\exp(-iH\tau)\ A\ \exp(iH t) = B$$
$$\textit{such that} \quad \exp(-iH\tau)\ A\ |\psi\rangle = B\ \exp(-iHt)\ |\psi\rangle.$$

A beautiful exact eigengate can be obtained when we have

$$[H, A] = iB,$$
$$[H, B] = -iA, \qquad (1)$$

because then

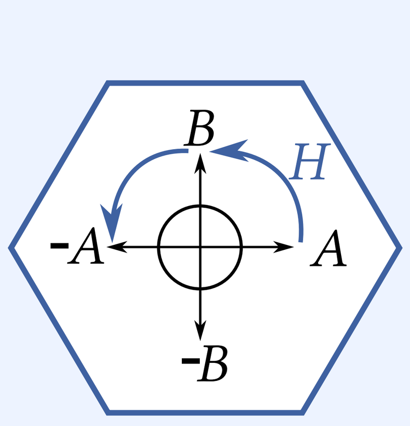
$$e^{-iHt} A e^{iHt} = A - it \underbrace{[H, A]}_{iB} + \frac{(-it)^2}{2!} \underbrace{[H, [H, A]]}_{A} + \ldots$$
$$= \sum_{\text{k even}} (-1)^{\frac{k}{2}} \frac{t^k}{k!} A + \sum_{\text{k odd}} (-1)^{\frac{k+1}{2}} \frac{t^k}{k!} B$$
$$= \cos(t)\ A\ +\ \sin(t)\ B.$$

By choosing $t = \pi/4$, we obtain our eigengate. Doubling the time, $t = \pi/2$, we *negate* all the eigenvalues of A.

Alternatively, one could adiabatically evolve $t \in (0, \frac{\pi}{2})$,

$$H(t) = \cos(t)\ A + \sin(t)\ B,$$

which has the useful property that energies remain the same at all times.



We may choose:

$$H \to \sigma^y$$
$$A \to \sigma^z \qquad [\sigma^y, \sigma^z] = 2i\sigma^x$$
$$B \to \sigma^x \qquad [\sigma^y, \sigma^x] = -2i\sigma^y$$

Hence $\exp(-i\sigma^y \frac{\pi}{4})$ is an eigengate between σ^z and σ^x.

The Pauli-matrices are a prototypical example.

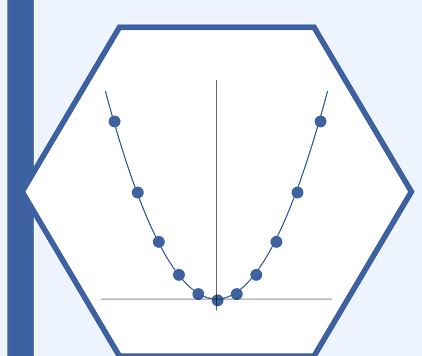
We re-interpret work by Polychronakos and Fahm, dating back to 1993, to feature an eigengate.

Another example derives from the work of Polychronakos, who considered the operators

$$A = \frac{1}{2} \sum_{j=1}^{N} x_j \sigma_j^z, \qquad B = \frac{1}{4} \sum_{j \neq k} w_{jk} \left(\sigma_j^x \sigma_k^y - \sigma_j^y \sigma_k^x \right),$$
$$w_{jk} = \frac{1}{x_j - x_k}.$$

The locations x_j of the qubits are given by the equilibrium positions of N particles with $1/r^2$ repulsion in a parabolic potential (a classical Colagero system):

$$V(x_1, \ldots, x_N) = \frac{1}{2} \sum_{j=1}^{N} x_j^2 + \sum_{j<k} \frac{1}{(x_j - x_k)^2}.$$



It turns out that we obtain our favorite commutation relations (Eq 1) with the Hamiltonian

$$H_{\mathrm{P}} = \sum_{j<k} h_{jk} P_{jk},$$

$$h_{jk} = \frac{1}{(x_j - x_k)^2}, \qquad P_{jk} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Alternatively, the Pauli matrices can form a **Hadamard gate**:

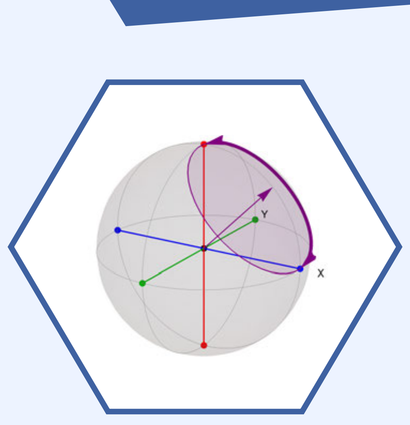
$$\tilde{H} = \frac{\sigma^z + \sigma^x}{\sqrt{2}} \qquad \sigma^z = \frac{\tilde{H} + \tilde{G}}{\sqrt{2}}$$
$$\tilde{G} = \frac{\sigma^z - \sigma^x}{\sqrt{2}} \qquad \sigma^x = \frac{\tilde{H} - \tilde{G}}{\sqrt{2}}$$

$$[\tilde{H}, \tilde{G}] = 2i\sigma^y \qquad [\tilde{H}, \sigma^y] = -2i\tilde{G}$$

such that

$$\sqrt{2}\ e^{-i\tilde{H}t} \sigma^z e^{i\tilde{H}t} = e^{-i\tilde{H}t} \left(\tilde{H} + \tilde{G} \right) e^{i\tilde{H}t}$$
$$= \tilde{H} + \cos(2t)\ \tilde{G} + \sin(2t)\ \sigma^y$$
$$= \sqrt{2}\ \sigma^x \qquad (\text{when } t = \pi/2).$$

Specifically for SU(2) relations, we may build a Hadamard gate from just A and B.



The Krawtchouk Chain mimics the Hamiltonian of a spin-s particle. Hence, we can rotate states around its Bloch Sphere, even for many-body states.

Recall that for a general spin-s particle, the spin matrices corresponding to a magnetic field in the Z- or X-direction are

$$S^z = \begin{pmatrix} s & & & \\ & s-1 & & \\ & & \ddots & \\ & & & -s \end{pmatrix}, \qquad S^x = \begin{pmatrix} 0 & b_s & & \\ b_s & 0 & \ddots & \\ & \ddots & \ddots & b_{-s+1} \\ & & b_{-s+1} & 0 \end{pmatrix},$$

$$b_j = \sqrt{(s+j)(s+1-j)}.$$

These have the same $\mathbf{su(2)}$ commutation relations as the Pauli matrices discussed before.

Now consider a chain of $N = 2s + 1$ qubits, where each qubit represent one of the spin states. These mimic the spin matrices above by choosing

$$H^z = \frac{1}{2} \sum_{j=-s}^{s} j\ \sigma^z$$
$$H^x = \frac{1}{2} \sum_{j=-s+1}^{s} b_j\ \sigma_j^+ \sigma_{j+1}^- + h.c..$$

As before, we readily obtain an *eigengate* by applying either the 'Hadamard' $H^z + H^x$, or the operator H^y that follows from the Y spin matrix. The many-body Hamiltonian H^x is known in literature as the **Krawtchouk chain**, whose eigenstates are highly entangled states with amplitudes described by Krawtchouk Polynomials.

Note that when multiple qubits are excited, the eigenstates of H_x are interferences of multiple S^x eigenstates. Because H^x can be mapped to a non-interacting fermionic system, the amplitudes are *Slater determinants* of the single-excitation amplitudes.

See also:

A. P. Polychronakos, *Lattice integrable systems of Haldane-Shastry type*, Phys. Rev. Lett. **70**, 2329 (1993)

H Frahm, *Spectrum of a spin chain with inverse square exchange*, J. Phys. A: Math. Gen. 26 L473 (1993)

KG & KS, *Many-body strategies for multi-qubit gates - quantum control through Krawtchouk chain dynamics*, ArXiv:1707.05144