Replica-nondiagonal solutions in the SYK model

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Introduction

The object of our study is the Sachdev-Ye-Kitaev model [1, 2, 3, 4], which is a theory of $N \gg 1$ interacting Majorana fermions in $0+1$ dimensions:

$$H = \frac{i^{q/2}}{q!} \sum_{i_1, i_2, \dots, i_q = 1}^{N} j_{i_1 i_2 \dots i_q} \psi_{i_1} \psi_{i_2} \dots \psi_{i_q}\,.$$

Here ψ_i are the Majorana fermions, and $j_{i_1 \dots i_q}$ are totally antisymmetric couplings randomized via the Gaussian distribution. To average over the disorder, one introduces replicas of the SYK. For example, the free energy is given by $-\beta F = \overline{\ln Z} = \lim_{M\to 0} \frac{\ln \overline{Z^M}}{M}$. After disorder averaging in $\overline{Z^M}$, one can introduce auxiliary fields G, Σ via the Hubbard-Stratonovich transformation such that $\Sigma_{\alpha\beta}(\tau,\tau')$ is a Lagrange multiplier which sets $G_{\alpha\beta}(\tau,\tau') \sim \frac{1}{N}\sum_i \psi_i^{\alpha}(\tau)\psi_i^{\beta}(\tau')$, and $\alpha, \beta = 1, \dots, M$. After integrating out the fermions, one obtains [3, 1, 4] (in Euclidean signature):

$$\overline{Z(\beta)^M} = \int DG D\Sigma \, \mathrm{Pf}[\delta_{\alpha\beta}\partial_\tau - \Sigma_{\alpha\beta}]^N \times \exp\left[-\frac{N}{2}\int_0^\beta \int_0^\beta d\tau_1 d\tau_2 \left(\Sigma_{\alpha\beta}(\tau_1,\tau_2) G_{\alpha\beta}(\tau_1,\tau_2) - \frac{J^2}{q} G_{\alpha\beta}(\tau_1,\tau_2)^q\right)\right]. \quad (1)$$

Note that the derivation includes steps which are not well-defined, which leads to the formally divergent path integral.

The goal of the work is the analytic study of replica-nondiagonal saddle points of the disorder-averaged free energy of the SYK model

- Construction of solutions of the saddle point equations using simple ansatz, in which the time dependence and replica dependence are factorized.
- Analysis of saddle point structure and contributions to the free energy of such solutions

Saddle point equations and the ansatz at finite M

We assume that the strong coupling is in place: $\beta J \gg 1 \Leftrightarrow \partial_\tau \to 0$.

The saddle points of the path integral are defined by the following equations:

$$\int d\tau' G_{\alpha\beta}(\tau,\tau')\Sigma_{\beta\gamma}(\tau',\tau'') = -\delta_{\alpha\gamma}\delta(\tau-\tau''); \qquad \Sigma_{\alpha\beta}(\tau,\tau') = J^2 G_{\alpha\beta}(\tau,\tau')^{q-1}. \quad (2)$$

We are going to study the solutions of saddle point equations using the particular ansatz, where the time and replica dependencies are factorized [10]:

$$G_{\alpha\beta}(\tau,\tau') = g(\tau,\tau') P_{\alpha\beta}. \quad (3)$$

$G_{\alpha\beta}(\tau,\tau') = -G_{\beta\alpha}(\tau',\tau) \Rightarrow$ we assume that $g(\tau,\tau') = -g(\tau',\tau)$ and $P_{\alpha\beta} = P_{\beta\alpha}$. G: Substituting (2) into (2), using the ansatz (3) and taking the diagonal component with $\alpha = \gamma$ we get the equation

$$J^2 \mathfrak{C} \int d\tau' g(\tau,\tau') g(\tau',\tau'')^{q-1} = -\delta(\tau - \tau''), \quad (4)$$

assuming that

$$\sum_\beta P_{\alpha\beta}^q = \mathfrak{C} = \text{const} \quad \forall \alpha, \quad (5)$$

Equation (4) up to normalization is the same equation as the equation for the replica-symmetric solution, so we can readily write down the solution (at finite temperature):

$$g(\tau,\tau') = \frac{b}{\mathfrak{C}^\Delta}\left(\frac{\pi}{\beta J}\right)^{2\Delta} \frac{\mathrm{sgn}(\tau-\tau')}{\left|\sin\frac{\pi}{\beta}(\tau-\tau')\right|^{2\Delta}}, \quad (6)$$

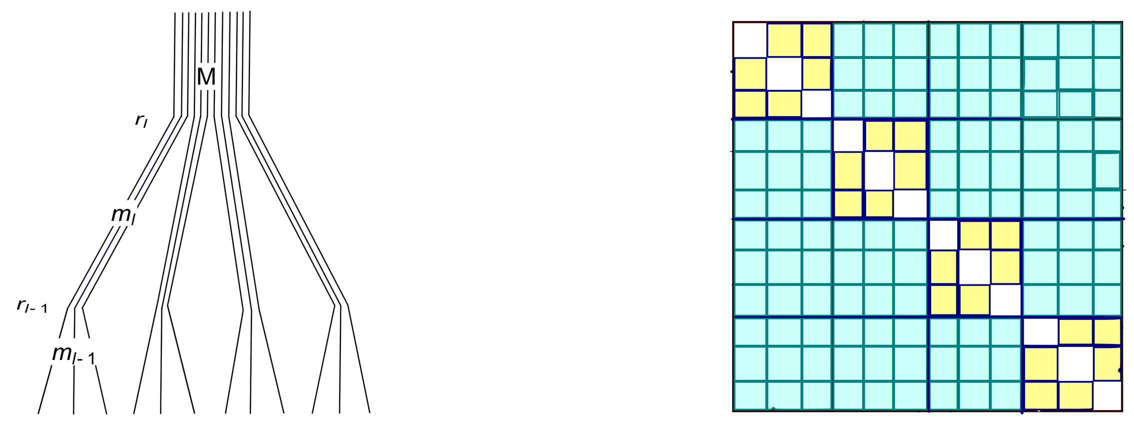
Now, when using this solution for g, the off-diagonal component gives equation for P:

$$\sum_\beta P_{\alpha\beta} P_{\beta\gamma}^{q-1} = 0, \quad \alpha \neq \gamma. \quad (7)$$

Our approach

1. Assume that the matrix P is of Parisi form [9].
2. Analytically continue to non-integer M and take the limit $M \to 0$ in the saddle point equation (7) and on-shell action
3. Construct the solutions of the resulting equations for the Parisi function
4. Compute the on-shell action, study the saddles

Parisi matrices



We define the Parisi matrix P in terms of the Parisi algebra generators $\mathfrak{I}_{m_i}\mathfrak{I}_{m_i} = I_{M/m_i} \otimes \mathfrak{J}_{m_i}$, where $(\mathfrak{J}_i)_{kj} = 1, \quad k, j = 1, \dots, m_i$. Here I_p is a unit matrix of dimension p. Then one expands as

$$P = \sum_{i=1,\, m_i \in \mathfrak{T}}^{l} a_i (\mathfrak{I}_{m_{i+1}} - \mathfrak{I}_{m_i}) + a_0 \mathfrak{I}_1 \quad (8)$$

. Using the algebra $\mathfrak{I}_{m_i}\mathfrak{I}_{m_j} = \mathfrak{I}_{m_j}\mathfrak{I}_{m_i} = m_i \mathfrak{I}_{m_j}$, for $i \le j$ we can calculate a product of two Parisi matrices $A = P = \sum_{i=1,\, m_i\in\mathfrak{T}}^{l} a_i(\mathfrak{I}_{m_{i+1}} - \mathfrak{I}_{m_i}) + a_0\mathfrak{I}_1$ and $B = P^{\circ(q-1)} = \sum_{i=1,\, m_i\in\mathfrak{T}}^{l} a_i^{q-1}(\mathfrak{I}_{m_{i+1}} - \mathfrak{I}_{m_i}) + a_0^{q-1}\mathfrak{I}_1$.

Thus we obtain the saddle point equations (7) with the constraint (5) in terms of Parisi variables:

$$\mathfrak{C} = a_0^q + \sum_{j=1}^{l} a_j^q (m_{j+1} - m_j); \quad (9)$$

$$0 = a_j a_0^{q-1} + a_0 a_j^{q-1} + \sum_{i<j}(a_i a_j^{q-1} + a_j a_i^{q-1})(m_{i+1} - m_i) - m_j a_j^q + \sum_{i>j-1} a_i^q (m_{i+1} - m_i). \quad (10)$$

On-shell action

To compute the on-shell action, we need the Pfaffian:

$$\mathrm{Pf}(-\hat{\Sigma}_{\alpha\beta}) = \int \prod_n \prod_\alpha d\tilde{\chi}_\alpha(\omega_n) \exp\left(-\frac{1}{2}\sum_n \tilde{\chi}_\alpha(\omega_n)\tilde{\Sigma}(\omega_n) P_{\alpha\beta}^{q-1} \tilde{\chi}_\beta(\omega_n)\right) = \prod_n (-\tilde{\Sigma}(\omega_n))^{\frac{M}{2}} \left[\det(P_{\alpha\beta}^{q-1})^{1/2}\right]^{d_F},$$

where $d_f \to \infty$.

Regularization: we impose hard cutoff on Matsubara frequencies in such a way that the IR limit is valid: $|\omega_n| \le J$, which means we have to set $d_f \sim \beta J$.

The on-shell action of the partition function (1) at finite M on factorized solutions is given by

$$\frac{2}{N} S_M = M(\mathfrak{s}_{RD} + \mathfrak{s}_{RND}), \quad \text{where} \quad (11)$$

$$\mathfrak{s}_{RD} = -\log \mathrm{Det}[-J^2 g_c(\tau,\tau')^{q-1}] + d_f\left(1 - \frac{1}{q}\right) J^2; \quad \mathfrak{s}_{RND} = -\frac{1}{M} d_f \log\det[\mathfrak{C}^{\Delta-1} P^{\circ(q-1)}]. \quad (12)$$

where we denoted $g_c(\tau,\tau') = \mathfrak{C}^\Delta g(\tau,\tau')$, and on the saddle point $\overline{Z^M} \sim e^{-S_M}$. Note that for now we work in the leading order of the strong coupling limit. **The contribution of the Parisi replica matrix to free energy is contained in $\mathfrak{s}_{RND}$.**

The $M \to 0$ limit

Analytic continuation to the Parisi function:

$$\sum_{i=1}^{n} \rho_i(m_{i+1} - m_i) \to \int_1^n \rho(v)\, dv\,; \qquad \sum_{i=j+1}^{n} \rho_j(m_{i+1} - m_i) \to \int_u^n \rho(v)\, dv\,.$$

Taking $M \to 0$, defining the average $\int_0^1 a^p(v)\,dv \equiv \langle a^p\rangle$, and fixing the scaling freedom by setting $a_0 = 1$, we arrive at the following equations:

$$\mathfrak{C} = 1 - \langle a^q \rangle\,. \quad (13)$$

$$0 = a(u)[1 - \langle a^{q-1}\rangle] + a^{q-1}(u)[1 - \langle a\rangle] - \int_0^u [a(v) - a(u)][a^{q-1}(v) - a^{q-1}(u)]\, dv\,, \quad (14)$$

where $u \in [0,\ 1]$. Thus, the dynamical variables are a_0 and $a(u)$. Defining the Parisi matrix $Q = \mathfrak{C}^{\Delta-1} P^{\circ(q-1)}$, the contribution to the free energy is computed as

$$\Delta F = F_{RND} - F_{RD} = \lim_{M\to 0} \frac{N}{2\beta} \mathrm{Re}\, \mathfrak{s}_3\Big|_{reg} = -\lim_{M\to 0} \frac{J N}{2} \frac{1}{M} \mathrm{Re}[\mathrm{tr} \log Q]\,. \quad (15)$$

We have taken the real part because the contribution from imaginary The tracelog is computed using the Parisi formula:

$$-\lim_{M\to 0}\frac{1}{M}\mathrm{tr}\ \log Q = -\log(q_0 - \langle q\rangle) - \frac{q(0)}{q_0 - \langle q\rangle} + \int_0^1 \frac{dv}{v^2} \log\frac{q_0 - \langle q\rangle - [q](v)}{q_0 - \langle q\rangle} \quad (16)$$

$$= -\frac{1-q}{q}\log\mathfrak{C} - \log(1 - \langle a^{q-1}\rangle) - \frac{a^{q-1}(0)}{1 - \langle a^{q-1}\rangle} + \int_0^1 \frac{dv}{v^2}\log\frac{1 - \langle a^{q-1}\rangle - [a^{q-1}](v)}{1 - \langle a^{q-1}\rangle}. \quad (17)$$

One-step replica symmetry breaking solutions

We restrict ourselves to the solutions for $a(u)$, which can be described by the one-step replica symmetry breaking ansatz, in analogy with the spin glass systems [9], such as the Sachdev-Ye model [7] (we fix $q = 4$):

$$a(u) = A_0 + A_1\theta(u - \mu)\,. \quad (18)$$

In this formula μ is a free parameter, to which we will refer as the breakpoint. We have obtained the following solutions (16 solutions total):

1. Replica-diagonal (paramagnetic) solution: $A_0 = 0$, $A_1 = 0$.
2. Replica-symmetric complex-valued solutions. They are shown by the pair of green points on the Fig.2A.

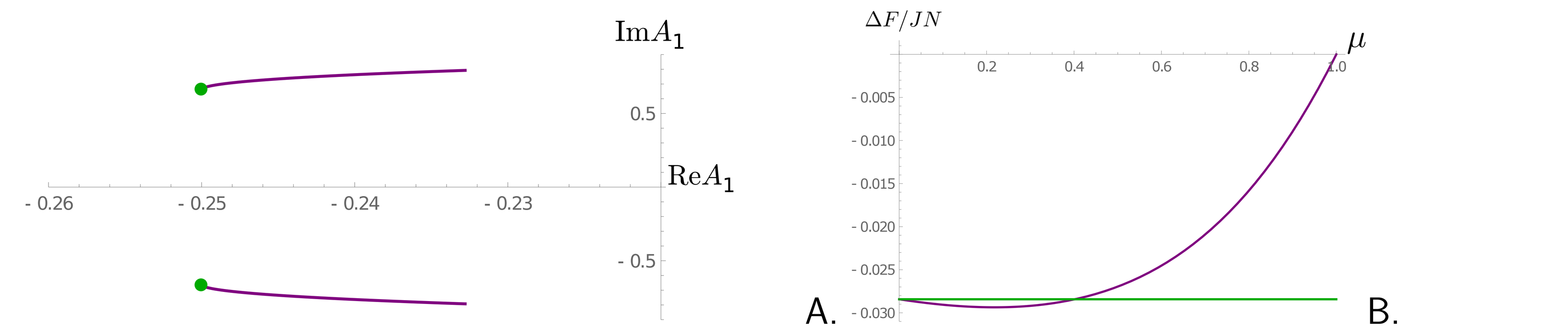


Figure: **A.** Trajectories of complex saddle points with $A_0 = 0$. **B.** The value of the free energy on these solutions. The green points show the location of replica-symmetric solution and the corresponding real part of the free energy density.

3. $A_0 = 0$, and A_1 is a solution of the equation $A_1^2 + 1 + A_1^3(\mu - 2) = 0$ which follows from (14). This equation has one real and two complex mutually conjugated solutions. These functions are represented graphically as purple lines on the above plot.

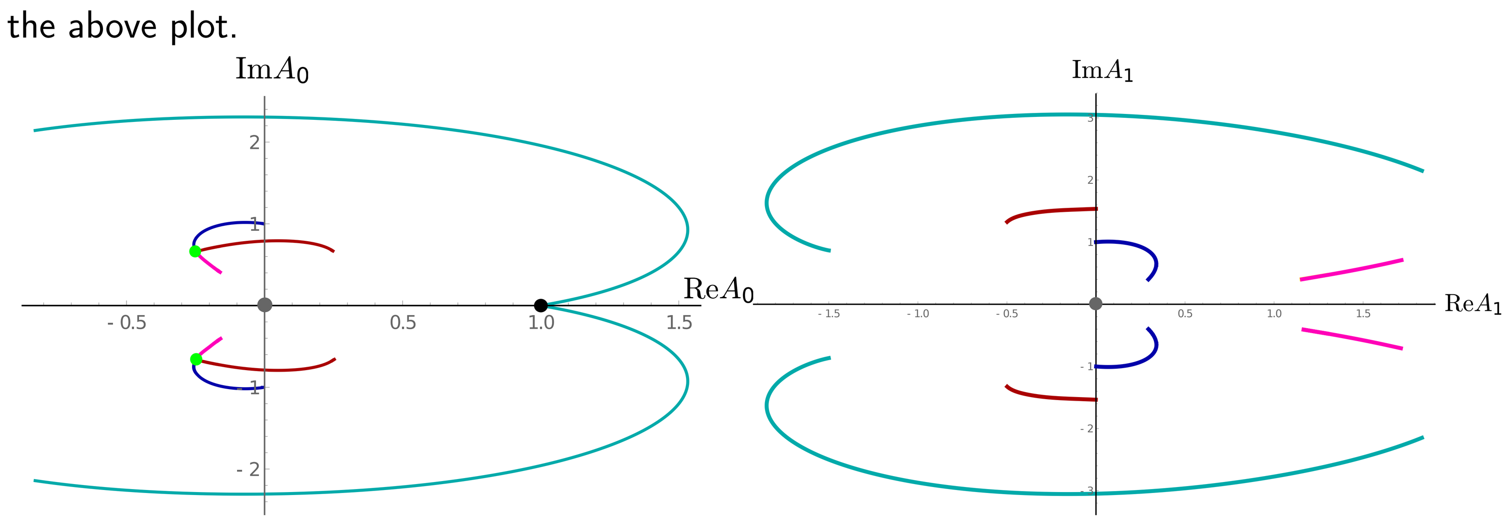


Figure: Trajectories of saddle points with non-zero A_0 and A_1, parametrized by μ on the complex plane.

4. Second group of RSB solutions is characterized by non-zero both A_0 and A_1.

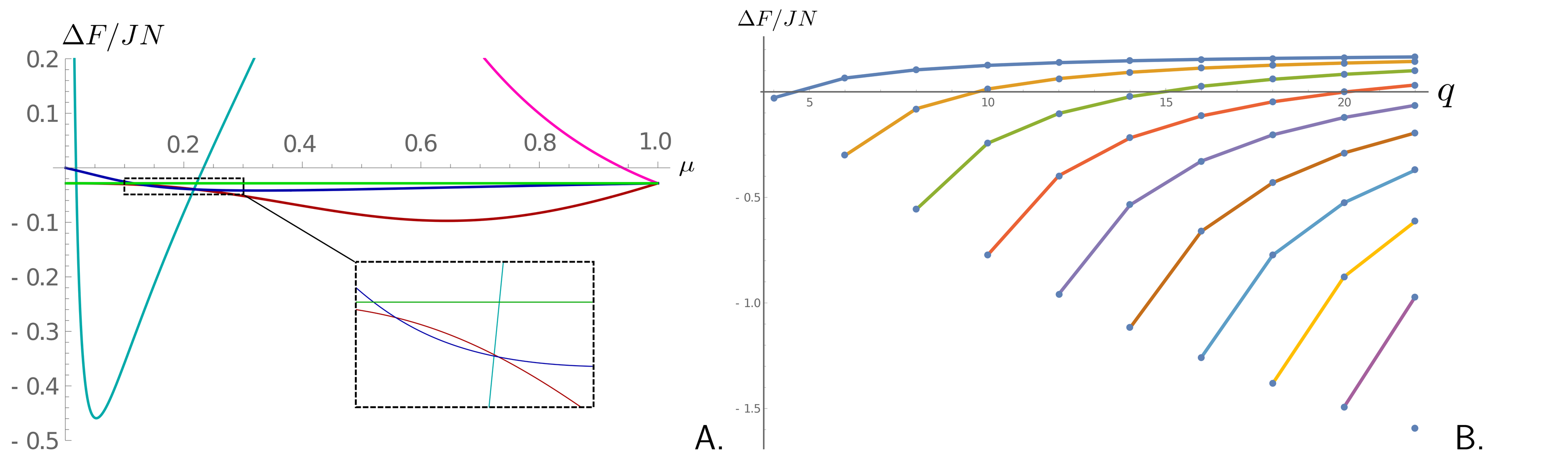


Figure: **A.** Free energy density on the solutions as function of μ. **B.** Free energy on the complex replica-symmetric saddles as a function of q.

Conclusions & outlook

- We analytically constructed a family of replica-nondiagonal solutions using the Parisi ansatz in SYK. It was done by solving the saddle point equations in the conformal limit after taking the limit $M \to 0$. The resulting integral equation (14) is turned into an algebraic when using the one-step RSB ansatz.
- The replica structure of the solutions contributes to the free energy. There are saddles, that correspond to replica symmetry breaking solution, with free energy lower than the replica-diagonal value.
- We also showed that leading non-conformal action is given by the sum of Schwarzian action over replicas:

$$I_{local} = -N\frac{\alpha_S}{J}\int_0^\beta \sum_\alpha \mathrm{Sch}\left(\tan\frac{\pi f_\alpha(\tau)}{\beta}, \tau\right) d\tau\,.$$

In the limit $M \to 0$ this gives the same contribution to the free energy as in the replica-diagonal case.

Open questions:

- Stability of the replica-nondiagonal saddles to fluctuations
- Commutativity of the replica limit and saddle point approximation
- Independence on the regularization and UV completion of the replica-nondiagonal solutions
- Holographic interpretation?

References

[1] A. Kitaev, talks at KITP in 2015: http://online.kitp.ucsb.edu/online/entangled15/kitaev/, http://online.kitp.ucsb.edu/online/entangled15/kitaev2/

[2] S. Sachdev and J. Ye, Phys. Rev. Lett. 70, 3339 (1993) [cond-mat/9212030].

[3] J. Maldacena and D. Stanford, Phys. Rev. D 94, no. 10, 106002 (2016) [arXiv:1604.07818 [hep-th]].

[4] A. Kitaev and S. J. Suh, arXiv:1711.08467 [hep-th].

[5] J. Polchinski and V. Rosenhaus, JHEP 1604, 001 (2016) [arXiv:1601.06768 [hep-th]].

[6] J. Maldacena, D. Stanford and Z. Yang, PTEP 2016, no. 12, 12C104 (2016) doi:10.1093/ptep/ptw124 [arXiv:1606.01857 [hep-th]].

[7] A. Georges, O. Parcollet and S. Sachdev, Phys. Rev. B 63 (Apr., 2001) 134406 [cond-mat/0009388].

[8] W. Fu and S. Sachdev, Phys. Rev. B 94, no. 3, 035135 (2016) [arXiv:1603.05246 [cond-mat.str-el]].

[9] M. Mezard and G. Parisi, "Replica field theory for random manifolds," LPTENS-90-28.

[10] A. Kamenev, talk at Steklov Mathematical Institute, March 2018

[11] I. Aref 'eva and I. Volovich, "Notes on the SYK model in real time," arXiv:1801.08118 [hep-th].