Non-Abelian Vortex Strings in supersymmetric gauge theories

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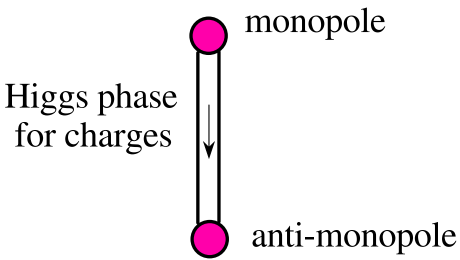
Introduction

Nambu, Mandelstam, 't Hooft and Polyakov 1970's:
Confinement is a dual Meissner effect upon condensation of monopoles.

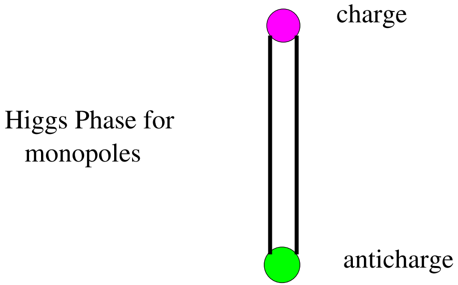
Electric charges condense $\rightarrow$ magnetic Abrikosov-Nielsen-Olesen flux tubes (strings) are formed $\rightarrow$ monopoles are confined



Nambu, Mandelstam, 't Hooft and Polyakov:

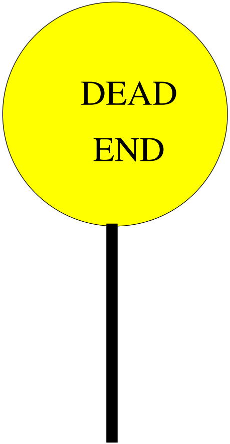
Dual Meissner effect:

Monopoles condense $\rightarrow$ electric Abrikosov-Nielsen-Olesen flux tubes are formed $\rightarrow$ electric charges are confined



$$V(R) = T\,R, \qquad T - \text{string tension}$$

No progress for many years...



QCD:

- No monopoles
- No confining strings
- Strong coupling

Breakthrough discovery come from supersymmetry.

Seiberg and Witten 1994 : Exact solution of

$\mathcal{N} = 2$ supersymmetric QCD.

Supersymmetric gauge theories can be considered as a ''theoretical laboratory" to develop insights in the dynamics of non-Abelian gauge theories.

Supersymmetric theories are "simplier" then real-world QCD

Many aspects are determined by exact solutions.

Example: $\mathcal{N} = 2$ Yang-Mills theory with gauge group $SU(2)$
The field content:
SU(2) gauge field A^a_μ,
$+$ adjoint complex scalar $=$ scalar gluon a^a, $a = 1, 2, 3$
$+$ fermions
Like Georgi-Glashow model
Adjoint scalar develops condensate $\rightarrow$ 't Hooft-Polyakov monopoles

Seiberg and Witten 1994 : Confinement in the monopole vacuum of $\mathcal{N} = 2$ QCD

Cascade gauge symmetry breaking:

- SU(N)$\rightarrow$ U(1)$^{N-1}$ condensate of adjoint scalars
  Example: SU(2)$\rightarrow$ U(1)
- U(1)$^{N-1} \rightarrow 0$ condensate of monopoles

At the last stage Abelian Abrikosov-Nielsen-Olesen flux tubes are formed.

Abelian confinement

In search for non-Abelian confinement non-Abelian strings were found in $\mathcal{N} = 2$ U(N) QCD

Hanany, Tong 2003

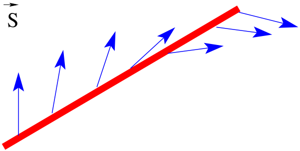
Auzzi, Bolognesi, Evslin, Konishi, Yung 2003

Shifman Yung 2004

Hanany Tong 2004

Non-Abelian string : Orientational zero modes

Rotation of color flux inside SU(N).



Abrikosov-Nielsen-Olesen strings

1. Higgs mechanism in Abelian Higgs model

$$S_{AH} = \int d^4x \left\{ -\frac{1}{4g^2} F_{\mu\nu}^2 + |\nabla_\mu q|^2 - \lambda(|q|^2 - \xi)^2 \right\}$$

where $\nabla_\mu q = (\partial_\mu - i n_e A_\mu)\, q$.
$U(1)$ gauge group is broken, $< q >= \sqrt{\xi}$, gauge field becomes massive

$$m_g = \sqrt{2} g n_e \sqrt{\xi}$$

The mass of the Higgs field is

$$m_H = 2\sqrt{\lambda}\sqrt{\xi}$$

Gauge phase is eaten
Number of degrees of freedom: Before 2+2=4 After 3+1=4

2. Abrikosov-Nielsen-Olesen vortices

Consider string-like solutions of equations of motion which depend only on x_i, $i = 1, 2$

$\pi_1(U(1)) = \mathbb{Z}$

At $r \to \infty$ we have

$$q \sim \sqrt{\xi} e^{in\alpha}, \qquad A_i \sim \frac{n}{n_e} \partial_i \alpha$$

where n is integer and r, α are polar coordinates in (x_1, x_2) plane.

$$\nabla_i q \sim in\partial_i \alpha - in_e \frac{n}{n_e} \partial_i \alpha \sim o(\frac{1}{r}), \qquad \int d^2x |\nabla_i q|^2 = \text{finite}$$

$$\Phi = \int d^2x \, F_3^* = \int_C dx_i A_i = \frac{n}{n_e} \int_C dx_i \partial_i \alpha = \frac{2\pi n}{n_e}, \quad F_3^* = \frac{1}{2} \varepsilon_{ij} F_{ij}.$$

Topological classes of fields A_i, q. Magnetic flux is quantized.

Ansatz for the string solution is

$$q = \phi(r)\, e^{in\alpha}, \qquad A_i = \frac{1}{n_e}\,\partial_i\alpha\,[n - f(r)]$$

with boudary conditions

$$\phi(0) = 0, \qquad \phi(\infty) = \sqrt{\xi},$$
$$f(0) = n, \qquad f(\infty) = 0$$

$$F_3^* = -\frac{1}{n_e\, r}\, f'(r), \quad \Phi = \int d^2x\, F_3^* = -\frac{2\pi}{n_e}\int_0^\infty dr\, f'(r) = \frac{2\pi\, f(0)}{n_e}$$

Singular gauge $U = e^{-in\alpha}$

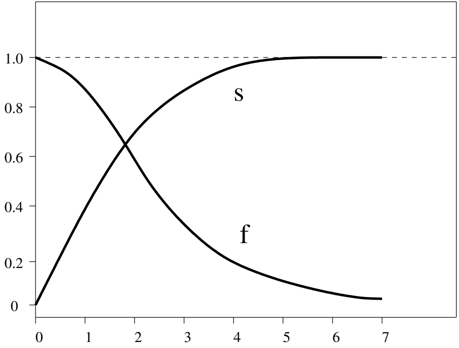
$$q = \phi(r), \qquad A_i = -\frac{1}{n_e}\,\partial_i\alpha\, f(r)$$

Equations of motion

$$\phi'' + \frac{\phi'}{r} - \frac{f^2\phi}{r^2} - m_H^2\,\frac{\phi(\phi^2 - \xi)}{2\xi} = 0$$

$$f'' - \frac{f'}{r} - \frac{m_g^2}{\xi}\,\phi^2\, f = 0$$

ANO string profile functions



Here $s = \phi/\sqrt{\xi}$.
At $r \to \infty$

$$f \sim e^{-m_g r}, \qquad (\phi - \sqrt{\xi}) \sim \sqrt{\xi}\, e^{-m_H r}$$

Superconductivity
Type I $m_H < m_g$, Type II $m_H > m_g$, BPS $m_H = m_g$,

BPS ANO strings in $\mathcal{N}=2$ supersymmetric QED

1. $\mathcal{N}=2$ QED

Field content:

Gauge multiplet A_{mu}, a + fermions λ_1 and λ_2

Matter multiplet q^A (charge =+1) , $\tilde{q}_A$ (charge =-1)

+ fermions ψ^A_α, $\tilde{\psi}_{\alpha A}$, $\alpha = 1, 2$, $A = 1, ..., N_f$

The bosonic part of the action

$$S = \int d^4x \left\{ -\frac{1}{4g^2} F^2_{\mu\nu} + \frac{1}{g^2} |\partial_\mu a|^2 + \bar{\nabla}_\mu \bar{q}_A \nabla_\mu q^A + \bar{\nabla}_\mu \tilde{q}_A \nabla_\mu \bar{\tilde{q}}^A \right.$$

$$- n_e^2 \frac{g^2}{2} \left(|q^A|^2 - |\tilde{q}_A|^2 - \xi \right)^2 - 2n_e^2 g^2 \left| \tilde{q}_A q^A \right|^2$$

$$\left. -\frac{1}{2}(|q^A|^2 + |\tilde{q}^A|^2) \left| 2n_e\, a + \sqrt{2} m_A \right|^2 \right\} ,$$

$$\nabla_\mu = \partial_\mu - i n_e A_\mu \,, \qquad \bar{\nabla}_\mu = \partial_\mu + i n_e A_\mu \,.$$

Consider the case $N_f = 1$.
The vacuum is given by

$$\langle a \rangle = -\frac{1}{n_e\sqrt{2}}\, m, \qquad \langle q \rangle = \sqrt{\xi}, \qquad \langle \tilde{q} \rangle = 0\,,$$

The spectrum:
One real component of field q is eaten up by the Higgs mechanism to become the third components of the massive photon. Three components of the massive photon, one remaining component of q and four real components of the fields $\tilde{q}$ and a

3 +1+2 +2 =8

A_μ q a $\tilde{q}$

form one long $\mathcal{N} = 2$ multiplet (8 boson states + 8 fermion states), with mass

$$m_\gamma^2 = 2 n_e^2\, g^2\, \xi.$$

2. BPS ANO string solution

Look for the string solution using the ansatz

$$a = -\frac{1}{n_e\sqrt{2}}\, m, \qquad \tilde{q} = 0$$

Then the action becomes

$$S = \int \mathrm{d}^4x \left\{ -\frac{1}{4g^2} F_{\mu\nu}^2 + |\nabla_\mu q|^2 - \frac{g^2}{2}\, n_e^2\, \left(|q|^2 - \xi\right)^2 \right\}$$

Here $m_H = m_g$. Assume again that A_i and q fields depend only on x_i, $i = 1, 2$ and write for the string tension the Bogomolny representation

$$T = \int d^2x \left\{ \left[\frac{1}{\sqrt{2}g} F_3^* + \frac{g}{\sqrt{2}}\, n_e \left(|q|^2 - \xi\right) \right]^2 + |\nabla_1\, q + i\nabla_2\, q|^2 + n_e\, \xi\, F_3^* \right\},$$

Bogomolny representation ensures that for the given winding number n the string solution (minimum of energy) has tension which is determined by the topological charge (magnetic flux)

$$T_n = 2\pi n \xi$$

and satisfies the first order equations

$$F_3^* + g n_e \left(|q|^2 - \xi \right) = 0\,,$$

$$(\nabla_1 + i\nabla_2) q = 0\,.$$

For the elementary $n = 1$ string the solution can be found using the standard ansatz

$$q(x) = \phi(r)\, e^{i\,\alpha}\ , \qquad A_i(x) = \frac{1}{n_e}\, \partial_i \alpha\, [1 - f(r)]$$

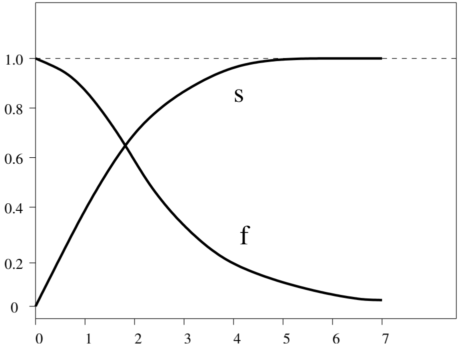
First order equations take the form

$$-\frac{1}{r}\,\frac{df}{dr} + n_e^2 g^2 \left(\phi^2 - \xi \right) = 0\ , \qquad r\,\frac{d\,\phi}{dr} - f\,\phi = 0$$

Boundary conditions

$$\phi(0) = 0, \qquad \phi(\infty) = \sqrt{\xi},$$
$$f(0) = 1, \qquad f(\infty) = 0$$

The profile functions for ANO BPS string can be found numerically



$\mathcal{N} = 2$ supersymmetric QCD in four dimensions

$\mathcal{N} = 2$ QCD with gauge group $U(N) = SU(N) \times U(1)$ and
$N_f = N$ flavors of fundamental matter – quarks

$+$

Fayet-Iliopoulos term of U(1) factor
The bosonic part of the action

$$S = \int d^4x \left[\frac{1}{4g_2^2} \left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2} \left(F_{\mu\nu}\right)^2 + \frac{1}{g_2^2} \left|D_\mu a^a\right|^2 + \frac{1}{g_1^2} \left|\partial_\mu a\right|^2 \right.$$

$$\left. + \left|\nabla_\mu q^A\right|^2 + \left|\nabla_\mu \bar{\tilde{q}}^A\right|^2 + V(q^A, \tilde{q}_A, a^a, a) \right] .$$

Here

$$\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu - iA^a_\mu T^a .$$

The potential is

$$
\begin{aligned}
V(q^A, \tilde{q}_A, a^a, a) &= \frac{g_2^2}{2}\left(\frac{i}{g_2^2} f^{abc}\bar{a}^b a^c + \bar{q}_A T^a q^A - \tilde{q}_A T^a \bar{\tilde{q}}^A\right)^2 \\
&+ \frac{g_1^2}{8}\left(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A - N\xi\right)^2 \\
&+ 2g_2^2 \left|\tilde{q}_A T^a q^A\right|^2 + \frac{g_1^2}{2}\left|\tilde{q}_A q^A\right|^2 \\
&+ \frac{1}{2}\sum_{A=1}^{N}\left\{\left|(a + \sqrt{2}m_A + 2T^a a^a)q^A\right|^2\right. \\
&+ \left.\left|(a + \sqrt{2}m_A + 2T^a a^a)\bar{\tilde{q}}^A\right|^2\right\} .
\end{aligned}
$$

Vacuum

$$\langle \frac{1}{2}\, a + T^a\, a^a \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & m_N \end{pmatrix},$$

For special choice

$$m_1 = m_2 = \dots = m_N$$

U(N) gauge group is unbroken.

$$\langle q^{kA} \rangle = \sqrt{\xi} \begin{pmatrix} 1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 1 \end{pmatrix}, \qquad \langle \bar{\tilde{q}}^{kA} \rangle = 0,$$

$$k = 1, \dots, N \qquad A = 1, \dots, N\,,$$

- Color-flavor locking
  Both gauge $U(N)$ and flavor $SU(N)$ are broken, however diagonal $SU(N)_{C+F}$ is unbroken

$$\langle q \rangle \to U \langle q \rangle U^{-1}$$

$$\langle a \rangle \to U \langle a \rangle U^{-1}$$

- Higgs phase $\Longrightarrow$ Gluons are massive

$$m_{SU(N)} = g_2 \sqrt{\xi}, \qquad m_{U(1)} = g_1 \sqrt{\frac{N}{2}\xi}$$

  Scalars a^a and a have the same masses. Quarks are combined with gauge bosons in long $\mathcal{N} = 2$ supermultiplets.

$$\underset{A_\mu}{(3} \; \underset{q}{+1} \; \underset{\tilde{q}}{+2} \; \underset{a}{+2})(N^2-1) = 8(N^2-1), \quad m_{SU(N)}$$

$$3 + 1 + 2 + 2 = 8, \quad m_{U(1)}$$

The theory is at weak coupling if we take

$$\sqrt{\xi} \gg \Lambda$$

$$\frac{8\pi^2}{g_2^2(\xi)} = N \log \frac{\sqrt{\xi}}{\Lambda} \gg 1$$

$$b = (2N - N_f) = N$$

$\mathbb{Z}_N$ strings

We look for string solutions which depend only on x_i, $i = 1, 2$
Example in $U(2) = U(1) \times SU(2)$
Abrikosov-Nielsen-Olesen (ANO) string:

$$q|_{r\to\infty} \sim \sqrt{\xi}\, e^{i\alpha} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_i \sim 2\,\partial_i \alpha, \qquad A_i^a = 0$$

Magnetic U(1) flux of ANO string is

$$\Phi = \int d^2x\, F_{12} = 4\pi$$

$\mathbb{Z}_2$ string:

$$q|_{r\to\infty} \sim \sqrt{\xi} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_i \sim \partial_i \alpha, \qquad A_i^3 \sim \partial_i \alpha$$

Magnetic U(1) flux of $\mathbb{Z}_2$ string is

$$\Phi = \int d^2x\, F_{12} = 2\pi$$

Here r and α are polar coordinates in the plane orthogonal to the string axis

We set a^a and a fields to their VEV's and put $\tilde{q} = 0$
The action of the model becomes

$$S = \int \mathrm{d}^4 x \left\{ -\frac{1}{4g_2^2} \left(F^a_{\mu\nu}\right)^2 - \frac{1}{4g_1^2} \left(F_{\mu\nu}\right)^2 \right.$$
$$\left. + |\nabla_\mu q^A|^2 - \frac{g_2^2}{2} \left(\bar{q}_A T^a q^A\right)^2 - \frac{g_1^2}{8} \left(|q^A|^2 - N\xi\right)^2 \right\}$$

Now we can write Bogomolny representation

$$T = \int d^2 x \left\{ \left[\frac{1}{\sqrt{2} g_2} F_3^{*a} + \frac{g_2}{\sqrt{2}} \left(\bar{q}_A T^a q^A\right) \right]^2 \right.$$
$$+ \left[\frac{1}{\sqrt{2} g_1} F_3^{*} + \frac{g_1}{2\sqrt{2}} \left(|q^A|^2 - N\xi\right) \right]^2$$
$$\left. + \left| \nabla_1 q^A + i \nabla_2 q^A \right|^2 + \frac{N}{2} \xi F_3^{*} \right\},$$

$$F_3^{*} = F_{12} \text{ and } F_3^{*a} = F^a_{12},$$

First order equations

$$F_3^* + \frac{g_1^2}{2}\left(\left|q^A\right|^2 - N\xi\right) = 0\,,$$
$$F_3^{*a} + g_2^2\left(\bar{q}_A T^a q^A\right) = 0\,,$$
$$(\nabla_1 + i\nabla_2)q^A = 0\,.$$

One can combine the Z_N center of SU(N) with the elements $\exp(2\pi ik/N)$ ∈U(1) to get a topologically stable string solution possessing both windings, in SU(N) and U(1).

$$\pi_1\left(\mathrm{SU}(N)\times\mathrm{U}(1)/Z_N\right) \neq 0\,.$$

This nontrivial topology amounts to selecting just one element of q, say, q^{11}, or q^{22}, etc, and make it wind

$$q_{\rm string} = \sqrt{\xi}\,\mathrm{diag}(1,1,...,e^{i\alpha})\,,\quad r\to\infty\,.$$

Elementary Z_N string solution

$$q = \begin{pmatrix} \phi_2(r) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \phi_2(r) & 0 \\ 0 & 0 & \dots & e^{i\alpha}\phi_1(r) \end{pmatrix},$$

$$A_i^{\mathrm{SU}(N)} = \frac{1}{N} \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix} (\partial_i \alpha) \left[-1 + f_{NA}(r)\right],$$

$$A_i = \frac{2}{N} (\partial_i \alpha) \left[1 - f(r)\right]$$

Magnetic U(1) flux of this Z_N string is

$$\int d^2x\, F_{12} = \frac{4\pi}{N}$$

First order equations

$$r\frac{d}{dr}\,\phi_1(r) - \frac{1}{N}\left(f(r) + (N-1)f_{NA}(r)\right)\phi_1(r) = 0\,,$$

$$r\frac{d}{dr}\,\phi_2(r) - \frac{1}{N}\left(f(r) - f_{NA}(r)\right)\phi_2(r) = 0\,,$$

$$-\frac{1}{r}\,\frac{d}{dr}f(r) + \frac{g_1^2 N}{4}\,\left[(N-1)\phi_2(r)^2 + \phi_1(r)^2 - N\xi\right] = 0\,,$$

$$-\frac{1}{r}\,\frac{d}{dr}f_{NA}(r) + \frac{g_2^2}{2}\,\left[\phi_1(r)^2 - \phi_2(r)^2\right] = 0\,.$$

Bogomolny representation gives tension of the elementary Z_N string

$$T = 2\pi\xi$$

Boundary conditions

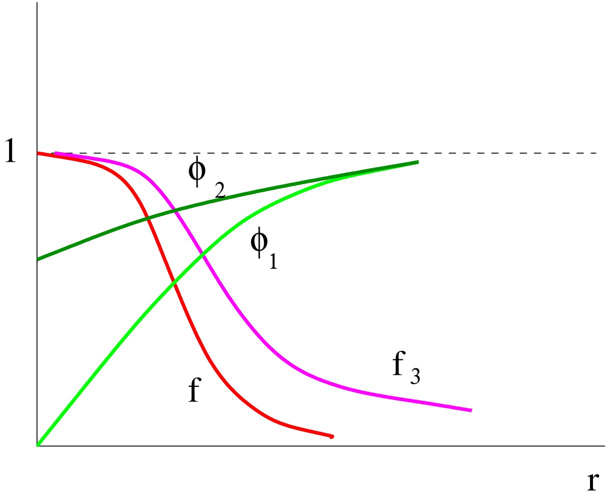
$$\phi_1(0) = 0,$$
$$f_{NA}(0) = 1, \quad f(0) = 1\,, \tag{1}$$

at $r = 0$, and

$$\phi_1(\infty) = \sqrt{\xi}, \quad \phi_2(\infty) = \sqrt{\xi}\,,$$
$$f_{NA}(\infty) = 0, \qquad f(\infty) = 0 \tag{2}$$

at $r = \infty$.

Profile functions of the string (for $N = 2$)



Non-Abelian strings

Vacuum is invariant with respect to $\mathrm{SU}(N)_{C+F}$ rotation while the solution is not. Therefore applying $\mathrm{SU}(N)_{C+F}$ rotation we get the infinite family of solutions.

1. Go to the singular gauge. 2. Apply $\mathrm{SU}(N)_{C+F}$ rotation.

$$q = U \begin{pmatrix} \phi_2(r) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \phi_2(r) & 0 \\ 0 & 0 & \dots & \phi_1(r) \end{pmatrix} U^{-1},$$

$$A_i^{\mathrm{SU}(N)} = \frac{1}{N} U \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix} U^{-1} \, (\partial_i \alpha) \, f_{NA}(r),$$

$$A_i = -\frac{2}{N} \, (\partial_i \alpha) \, f(r).$$

Z_N solution breaks $SU(N)_{C+F}$ down to $SU(N-1)\times U(1)$
Thus the orientational moduli space is

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1)\times\mathrm{U}(1)}\sim\mathrm{CP}(N-1)$$

Matrix U can be parametrized

$$\frac{1}{N}\left\{U\begin{pmatrix}1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots\\ 0 & \dots & 1 & 0\\ 0 & 0 & \dots & -(N-1)\end{pmatrix}U^{-1}\right\}_p^l=-n^l n_p^*+\frac{1}{N}\delta_p^l\ ,$$

with

$$n_l^* n^l=1$$

The number of parameters

$$N^2 - 1 - (N-1)^2 = 2(N-1)$$

Then the solution for the non-Abelian string takes the form

$$q = \frac{1}{N}[(N-1)\phi_2 + \phi_1] + (\phi_1 - \phi_2)\left(n \cdot n^* - \frac{1}{N}\right),$$

$$A_i^{\mathrm{SU}(N)} = \left(n \cdot n^* - \frac{1}{N}\right)\varepsilon_{ij}\,\frac{x_j}{r^2}\,f_{NA}(r),$$

$$A_i = \frac{2}{N}\varepsilon_{ij}\,\frac{x_j}{r^2}\,f(r),$$

$CP(N)$ model on the string

String moduli: x_{0i}, $i = 1, 2$ and n^l, $l = 1, ..., N$

Make them t, z-dependent. Translational moduli decouple. Consider orientational moduli.

Substitute the string solution into 4D action.

We have to switch on gauge components A_k, $k = 0, 3$. Use the ansatz

$$A_k^{\mathrm{SU}(N)} = -i \left[\partial_k n \cdot n^* - n \cdot \partial_k n^* - 2n \cdot n^*(n^* \partial_k n)\right] \rho(r)$$

This gives

$$\begin{aligned} F_{ki}^{\mathrm{SU}(N)} &= (\partial_k n \cdot n^* + n \cdot \partial_k n^*)\, \varepsilon_{ij} \frac{x_j}{r^2} f_{NA} [1 - \rho(r)] \\ &+ i [\partial_k n \cdot n^* - n \cdot \partial_k n^* - 2n \cdot n^*(n^* \partial_k n)] \frac{x_i}{r} \frac{d\rho(r)}{dr} . \end{aligned}$$

To have a finite contribution from the term $\mathrm{Tr}\, F_{ki}^2$ in the action we impose the constraint

$$\rho(0) = 1 \qquad \rho(\infty) = 0$$

Combining with contribution from quark kinetic terms we get 2D $CP(N-1)$ model

$$S^{(1+1)} = \beta \int dt\, dz \, \left\{ (\partial_k \, n^* \partial_k \, n) + (n^* \partial_k \, n)^2 \right\}$$

with inverse coupling β

$$\beta = \frac{4\pi}{g_2^2} \, I ,$$

where

$$I = \int_0^\infty rdr \left\{ \left(\frac{d}{dr} \rho(r) \right)^2 + \frac{1}{r^2} \, f_{NA}^2 \, (1-\rho)^2 \right.$$

$$\left. + \; g_2^2 \left[\frac{\rho^2}{2} \left(\phi_1^2 + \phi_2^2 \right) + (1-\rho)\left(\phi_2 - \phi_1 \right)^2 \right] \right\} .$$

Minimizing with respect to ρ we get second order equation for ρ

$$-\frac{d^2}{dr^2}\,\rho-\frac{1}{r}\,\frac{d}{dr}\,\rho-\frac{1}{r^2}\,f_{NA}^2\,(1-\rho)+\frac{g_2^2}{2}\left(\phi_1^2+\phi_2^2\right)\rho-\frac{g_2^2}{2}\left(\phi_1-\phi_2\right)^2=0$$

The solution is

$$\rho = 1 - \frac{\phi_1}{\phi_2}$$

Then

$$I = 1, \qquad \beta = \frac{4\pi}{g_2^2}$$

The two-dimensional coupling is determined by the four-dimensional non-Abelian coupling.

The two-dimensional coupling is determined by the four-dimensional non-Abelian coupling. This relation is obtained at the classical level. In quantum theory both couplings run. What is the scale where this relation imposed?
The two-dimensional CP($N-1$) model is an effective low-energy theory appropriate for the description of internal string dynamics at low energies, lower than the inverse thickness of the string which is given by the masses of the gauge/quark multiplets

$$m_{SU(N)} = g_2 \sqrt{\xi}$$

Thus, the parameter $m_{SU(N)}$ plays the role of a physical ultraviolet (UV) cutoff of the world sheet sigma model. This is the scale at which the relation beween couplings holds. Below this scale, the coupling β runs according to its two-dimensional renormalization-group flow

$$2\pi\beta = N \ln \frac{m_{SU(N)}}{\Lambda_\sigma}, \qquad \frac{8\pi^2}{g_2^2(\xi)} = N \log \frac{m_{SU(N)}}{\Lambda_{\mathrm{SU}(N)}}$$

Equating two couplings we get

$$\Lambda_\sigma = \Lambda_{\mathrm{SU}(N)}$$

$CP(N-1)$ model is a low energy effective theory. There are infinite series of higher derivative corrections in powers of

$$\frac{\partial}{m_{SU(N)}}$$

to the action of $CP(N-1)$ model.

Example in U(2)

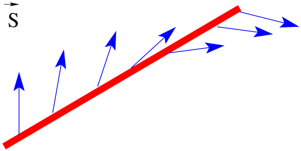
$$CP(1) = O(3)$$

We have two dimensional $O(3)$ sigma model living on the string world sheet.

$$S_{(1+1)} = \frac{\beta}{4}\int dt\, dz\, (\partial_k \vec{S})^2, \qquad \vec{S}^2 = 1$$

where

$$S^a = -n^* \tau^a n, \qquad a = 1, 2, 3$$



Gauge theory formulation of $CP(N-1)$ model

Witten 1979:

CP$(N-1)$ == Higgs branch of U(1) gauge theory

The bosonic part of the action is

$$S_{CP(N-1)} = \int d^2x \left\{ |\nabla_k n^l|^2 - \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 + \frac{1}{2e^2} D^2 \right. \\ \left. - \; 2|\sigma|^2 |n^l|^2 + D(|n^l|^2 - \beta) \right\},$$

Condition

$$|n^l|^2 = \beta\,,$$

imposed in the limit $e^2 \to \infty$

Gauge field can be eliminated:

$$A_k = -\frac{i}{2\beta}\left(\bar{n}_l \partial_k n^l - \partial_k \bar{n}_l\, n^l\right), \qquad \sigma = 0$$

Number of degrees of freedom $= 2N - 1 - 1 = 2(N-1)$

Our string is BPS $\Rightarrow$ $\mathcal{N} = (2,2)$ supersymmetric $CP(N-1)$ model

Large N solution of $CP(N-1)$ model

Witten 1979

Solved at large N both $\mathcal{N}=(2,2)$ and non-SUSY $CP(N-1)$ models.

At large N we integrate out fields n^l and their fermion superpartners

$$\left[\det\left(-\partial_k^2 - D + 2|\sigma|^2\right)\right]^{-N}\left[\det\left(-\partial_k^2 + 2|\sigma|^2\right)\right]^{N},$$

We get

$$-\frac{N}{4\pi}\left\{\left(-D+2|\sigma|^2\right)\left[\ln\frac{M_{\rm uv}^2}{-D+2|\sigma|^2}+1\right]-2|\sigma|^2\left[\ln\frac{M_{\rm uv}^2}{2|\sigma|^2}+1\right]\right\}$$

The scale Λ_σ is defined by writing the bare coupling as

$$\beta_0 = \frac{N}{4\pi}\ln\frac{M_{\rm uv}^2}{\Lambda_\sigma^2}$$

in the term $-D\beta_0$ in the action.

We get

$$V_{eff} = \int d^2x \, \frac{N}{4\pi} \left\{ - \left(-D + 2|\sigma|^2\right) \log \frac{-D + 2|\sigma|^2}{\Lambda_\sigma^2} - D + 2|\sigma|^2 \log \frac{2|\sigma|^2}{\Lambda_\sigma^2} \right\},$$

Minimizing this potential we get equations

$$2\beta_{\rm ren} = \frac{N}{4\pi} \log \frac{-D + 2|\sigma|^2}{\Lambda_\sigma^2} = 0 \qquad \to \qquad \langle |n^l|^2 \rangle = 0$$

$$\log \frac{-D + 2|\sigma|^2}{2|\sigma|^2} = 0$$

Solution:

$$2|\sigma|^2 = \Lambda_\sigma^2$$

$$D = 0.$$

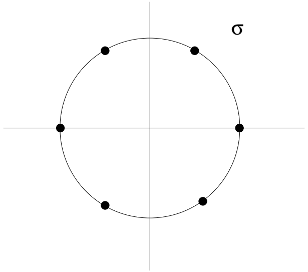
The model has U(1) axial symmetry which is broken by the chiral anomaly down to discrete subgroup Z_{2N} (*Witten 1979*). The field σ transforms under this symmetry as

$$\sigma \to e^{\frac{2\pi k}{N}\, i}\sigma, \qquad k = 1, ..., N-1.$$

Z_{2N} symmetry is spontaneously broken by the condensation of σ down to Z_2,

$$\sqrt{2}\langle\sigma\rangle = \Lambda\, e^{\frac{2\pi k}{N}\, i} \qquad k = 0, ..., N-1.$$

There are N strictly degenerate vacua



Classically n^l develop VEV, $\langle |n|^2 \rangle = \beta$

There are $2(N-1)$ massless Goldstone states.

In quantum theory this does not happen

$SU(N)_{C+F}$ global symmetry is unbroken

Mass gap $\sim \Lambda_{CP}$; no massless states ($\langle |n|^2 \rangle = 0$)

Kinks (domain walls) interpolating between different vacua.

Kink masses are nonzero

Kink sizes are stabilized in quantum regime, $\sim \Lambda_\sigma^{-1}$

Unequal quark masses

N quantum vacua of $CP(N-1)$ model and N $\mathbb{Z}_N$ strings?

Introduce quark mass differences, This breaks $SU(N)_{C+F}$ down to $U(1)^{N-1}$

Consider U(2) $\mathcal{N}=2$ QCD for simplicity. The string solution reduces to

$$
\begin{aligned}
q &= U \begin{pmatrix} \phi_2(r) & 0 \\ 0 & \phi_1(r) \end{pmatrix} U^{-1}\,, \\
A_i^a(x) &= -S^a\,\varepsilon_{ij}\,\frac{x_j}{r^2}\,f_{NA}(r)\,, \\
A_i(x) &= \varepsilon_{ij}\,\frac{x_j}{r^2}\,f(r)\,, \qquad S^a = -n^*\tau^a n\,.
\end{aligned}
$$

At large r the field a^a tends to its VEV aligned along the third axis in the color space,

$$
\langle a^3 \rangle = -\frac{\Delta m}{\sqrt{2}}, \quad \Delta m = m_1 - m_2,
$$

The ansatz for the adjoint scalar

$$a^a = -\frac{\Delta m}{\sqrt{2}} \, \left[\delta^{a3}\, b + S^a\, S^3\,(1-b)\right]$$

with boundary conditions

$$b(\infty) = 1\,, \qquad b(0) = 0$$

This gives the potential

$$V_{\mathrm{CP}(1)} = \gamma \int d^2x \, \frac{\Delta m^2}{2} \left(1 - S_3^2\right)\, ,$$

where

$$\gamma = \frac{2\pi}{g_2^2} \int_0^\infty r\, dr \left\{ \left(\frac{d}{dr} b(r)\right)^2 + \frac{1}{r^2}\, f_{NA}^2\, b^2 + \right.$$

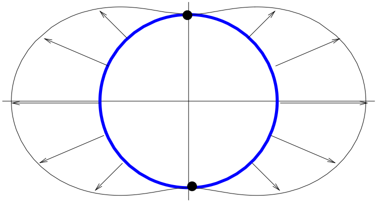
$$\left. + \; g_2^2 \left[\frac{1}{2}\,(1-b)^2 \, \left(\phi_1^2 + \phi_2^2\right) + b\, \left(\phi_1 - \phi_2\right)^2\right]\right\}\, .$$

Minimization with respect to $b(r)$ gives

$$b(r) = 1 - \rho(r) = \frac{\phi_1}{\phi_2}(r) \qquad \gamma = I \times \frac{2\pi}{g_2^2} = \frac{2\pi}{g_2^2}$$

Finaly we get

$$S_{\mathrm{CP}(1)} = \frac{\beta}{2} \int d^2x \left\{ \frac{1}{2} \left(\partial_k S^a\right)^2 + \frac{|\Delta m|^2}{2} \left(1 - S_3^2\right) \right\}$$



For arbitrary N in the GLSM description we have

$$S_{CP(N-1)} = \int d^2x \left\{ \left|\nabla_\alpha n^l\right|^2 - \frac{1}{4e^2} F_{\alpha\beta}^2 + \frac{1}{e^2} \, |\partial_\alpha \sigma|^2 \right.$$
$$\left. - \left|\sqrt{2}\sigma + m_l\right|^2 \left|n^l\right|^2 - \frac{e^2}{2} \left(|n^l|^2 - \beta\right)^2 \right\} + \text{fermions}$$

Classically at large mass differences N vacua are given by

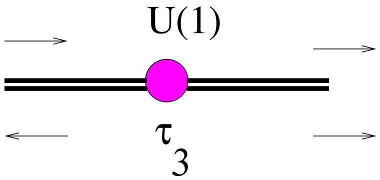
$$\langle n^l \rangle = \delta^{l l_0}, \qquad \langle \sqrt{2}\sigma \rangle = -m_{l_0}, \qquad l_0 = 1, ..., N$$

$\mathbb{Z}_N$ strings.

Confined monopoles

Higgs phase for quarks $\Longrightarrow$ confinement of monopoles
Elementary monopoles – junctions of two different strings
Example in $U(2)$

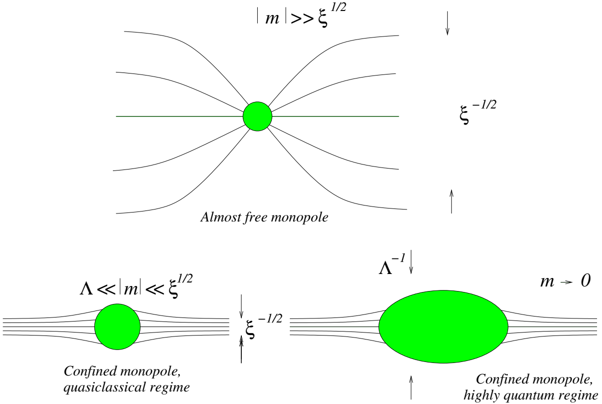
monopole



string flux $= \int dx_i \, A_i = 2\pi \, n \cdot n^*, \qquad n^l = \delta^{ll_0}$

monopole flux

$= \text{string flux}_{l_0}$ - $\text{string flux}_{l_0+1} = 4\pi \times \mathrm{diag}\frac{1}{2}\{..., 1, -1, ...\}$

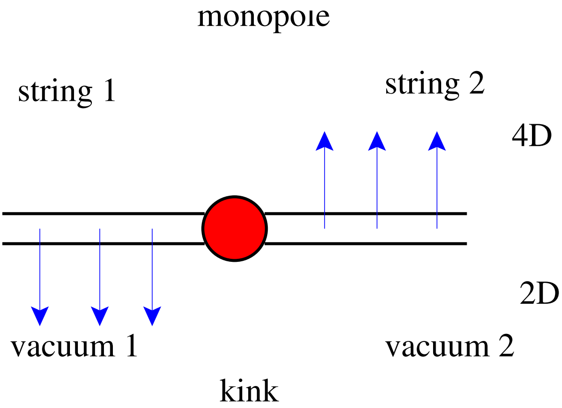


In 2D $CP(N-1)$ model on the string we have

N vacua $=$ N Z_N strings

and kinks interpolating between these vacua

Kinks = confined monopoles



The first-order equations for the string junction

String junction is 1/4-BPS

Consider U($N=2$) theory. $\Lambda_{CP} \ll |\Delta m| \ll \sqrt{\xi}$

Bogomolny representation

$$
\begin{aligned}
E \;=\; & \int d^3x \left\{ \left[\frac{1}{\sqrt{2}g_2} F_3^{*a} + \frac{g_2}{2\sqrt{2}} \left(\bar{q}_A \tau^a q^A \right) + \frac{1}{g_2} D_3 a^a \right]^2 \right. \\
+ & \left[\frac{1}{\sqrt{2}g_1} F_3^{*} + \frac{g_1}{2\sqrt{2}} \left(|q^A|^2 - 2\xi \right) + \frac{1}{g_1} \partial_3 a \right]^2 \\
+ & \frac{1}{g_2^2} \left| \frac{1}{\sqrt{2}} (F_1^{*a} + iF_2^{*a}) + (D_1 + iD_2) a^a \right|^2 \\
+ & \left| \nabla_1 \, q^A + i \nabla_2 \, q^A \right|^2 \\
+ & \left. \left| \nabla_3 q^A + \frac{1}{\sqrt{2}} \left(a^a \tau^a + a + \sqrt{2} m_A \right) q^A \right|^2 \right\} + E_{\mathrm{surface}}
\end{aligned}
$$

Surface terms

$$E_{\rm surface} = \xi \int d^3x F_3^* - \sqrt{2}\,\frac{\langle a^a \rangle}{g_2^2} \int dS_n\, F_n^{*a}$$

First order equations

$$\begin{aligned}
&F_1^{*a} + iF_2^{*a} + \sqrt{2}(D_1 + iD_2)a^a = 0\,, \\
&F_3^* + \frac{g_1^2}{2}\left(\left|q^A\right|^2 - 2\xi\right) + \sqrt{2}\,\partial_3 a = 0\,, \\
&F_3^{*a} + \frac{g_2^2}{2}\left(\bar{q}_A \tau^a q^A\right) + \sqrt{2}\,D_3 a^a = 0\,, \\
&\nabla_3 q^A = -\frac{1}{\sqrt{2}}\left(a^a \tau^a + a + \sqrt{2} m_A\right) q^A\,, \\
&(\nabla_1 + i\nabla_2) q^A = 0\,.
\end{aligned}$$

Ansatz for solution: String solution with z-dependence given by function $S^a(z)$

$$S^a(-\infty) = (0,0,1), \qquad S^a(\infty) = (0,0,-1)$$

First order equations are satisfied if

$$\partial_3 S^a = \Delta m \left(\delta^{a3} - S^a S^3\right), \qquad \Delta m = m_1 - m_2$$

This equation is the first order equation for kink in $O(3)$ sigma model

$$E = \frac{\beta}{4} \int dz \left\{ \left| \partial_z S^a - \Delta m \left(\delta^{a3} - S^a S^3\right) \right|^2 + 2 \Delta m \, \partial_z S^3 \right\}$$

$$M_{\rm kink} = \beta \, \Delta m, \qquad M_M = \frac{4\pi}{g_2^2} \Delta m$$

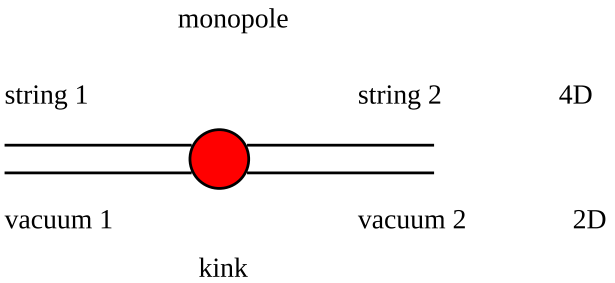
Since

$$\beta = \frac{4\pi}{g_2^2} \quad \to \quad M_M = M_{\rm kink}$$

2D-4D correspondence

Kinks = confined monopoles

$\mathcal{N} = (2,2)$ model :



$$M^{kink} = M^{\rm monopole}$$

independent on ξ.

Kink masses

$$M_{ll'}^{\rm BPS} = 2\left|\mathcal{W}_{\rm CP}(\sigma_{p'}) - \mathcal{W}_{\rm CP}(\sigma_l)\right| \,, \qquad l, l' = 1, ..., N$$

Compare with monopole masses

$$M_{ll'}^{\rm monopole} = \left| \frac{\sqrt{2}}{2\pi i} \oint_{\beta_{ll'}} d\lambda_{SW} \right|, \qquad l, l' = 1, ...N$$

$$M_{ll'}^{\rm monopole} = M_{ll'}^{\rm kink}, \qquad l, l' = 1, ..., N$$

Dorey 1998
Shifman Yung 2004
Hanany Tong 2004

Example in U(2)

U(2) gauge theory with $N_f = 2$

Exact formula for the kink mass

$$M^{\rm kink}_{r=N} = \left| \frac{1}{2\pi} \left\{ \Delta m \ln \frac{\Delta m + \sqrt{\Delta m^2 + 4\Lambda_{CP}^2}}{\Delta m - \sqrt{\Delta m^2 + 4\Lambda_{CP}^2}} - 2\sqrt{\Delta m^2 + 4\Lambda_{CP}^2} \right\} \right| ,$$

where $\Delta m = m_1 - m_2$

Confined monopoles = kinks are stabilized by quantum (non-perturbative) effects in $CP(N-1)$ model on the string worldsheet

Consider non-Abelian regime $(m_A - m_B) \to 0$

Classical picture

$$M_M = \frac{4\pi(m_{\ell_0+1} - m_{\ell_0})}{g_2^2} \to 0$$

monopole size $\sim \Delta m^{-1} \to \infty$

Classically monopole disappear

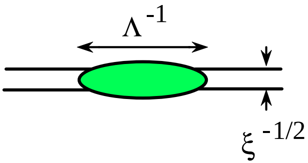
Quantum picture

$SU(N)_{C+F}$ global symmetry is unbroken

Mass gap $\sim \Lambda_{CP}$ no massless states $(\langle |n|^2 \rangle = 0)$

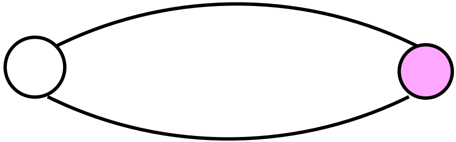
$$M_{\rm monopole} = M_{\rm kink} \sim \Lambda_{CP}$$

$$\text{monopole size} \sim \Lambda_{CP}^{-1}$$



Physical picture of monopole confinement

Monopole-antimonopole meson.



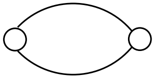
Witten 1989
kink $\sim n^l$ at strong coupling
Monopole (anti-monopole) $=$ kink (anti-kink) is in the fundamental (anti-fundamental) representation of global "flavor" $\mathrm{SU}(N)_{C+F}$

Baryons

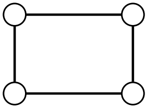
$$\Phi^{\mathrm{SU}(N)} = \int d^2x\, F_3^{*\mathrm{SU}(N)} = 2\pi \left(n \cdot n^* - \frac{1}{N} \right), \qquad n^l = \delta^{l l_0}$$

$$\sum_{l_0} \Phi_{l_0}^{\mathrm{SU}(N)} = 0$$

Therefore N different strings can form a closed configuration



a

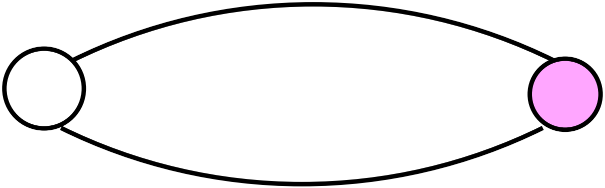


b

Instead-of-confinement phase

Meson

Constituent quark = monopole



At weak coupling these mesons are heavy and decay into screened quarks and gluons

What about strong coupling?

Curves (walls) of marginal stability in 2D

Example in CP(1)

$$Z_{\rm kink}^{\rm BPS} = m_D T + i\Delta m\, q, \qquad M_{\rm kink}^{\rm BPS} = |Z_{\rm kink}^{\rm BPS}|$$

T is the topologikal charge $T = 0, \pm 1$,
q is the global charge; $\mathrm{SU}(2)_{C+F} \to \mathrm{U}(1)$, $q = \pm\frac{1}{2}, \pm 1, \ldots$
Decay 3$\to$ 1+2,

$$T_3 = T_1 + T_2, \qquad q_3 = q_1 + q_2, \qquad Z_3 = Z_1 + Z_2$$

Curve of marginal stabilility

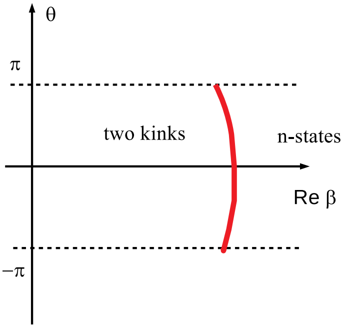
$$\mathrm{Re}\frac{m_D}{\Delta m} = 0$$

In particular, perturbative state with $T_3 = 0$, $q_3 = 1$ decay at kink $T_1 = 1$, $q_1 = \frac{1}{2}$ and antikink $T_2 = -1$, $q_2 = \frac{1}{2}$ at

$$\mathrm{Re}\frac{Z_1}{\Delta m} = 0$$

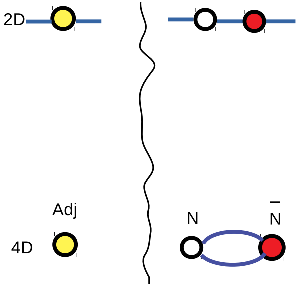
Curves (walls) of marginal stability

$$\beta = \mathrm{Re}\ \beta + i\frac{\theta_{2D}}{2\pi}, \qquad 2\pi\beta = 2\log\left(\frac{m_1 - m_2}{\Lambda_{CP}}\right)$$



Weak coupling   Strong coupling

perturbative state   kink   anti-kink



quark or gluon   monopole   anti-monopole

Question: Does these monopole-antimonopole mesons look like mesons in QCD?

- Correct flavor quantum numbers (adjoint + singlet)
- Lie on Regge tragectories

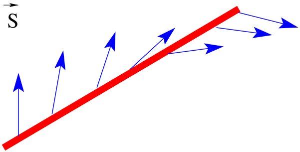
Instead-of-confinement phase is a new phase of asymptotically free non-Abelian gauge theories

besides Higgs and confinement phases known previously

Looks very close to what we observe in the real-world QCD

constituent quark = monopole

From non-Abelian vortices to critical superstrings



Shifman and Yung, 2015 Idea:

Non-Abelian vortex string has more moduli then Abrikosov-Nielsen-Olesen (ANO) vortex string.

It has translational + orientaional and size moduli: $x_\mu^{(0)}(\sigma,\tau)$ and $n^l(\sigma,\tau)$, $\rho^k(\sigma,\tau)$

We can fulfill the criticality condition: $4+6=10$

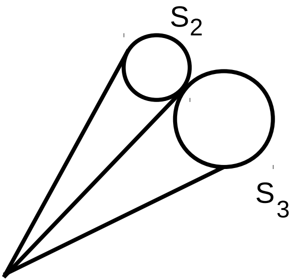
- The solitonic non-Abelian vortex have six orientational and size moduli, which, together with four translational moduli, form a ten-dimensional space ($N=2$, $N_f=4$).
- For $N_f=2N$ 2D world sheet theory on the string is conformal.

For U($N = 2$) gauge group and $N_f = 2$ the world sheet theory is 2D $O(3)$ sigma model, $O(3) = CP(1)$

For $N = 2$ and $N_f = 4$ the world sheet theory is weighted $CP(2,2)$ model.

The target space of the weighted $CP(2,2)$ model is a non-compact Calabi-Yau manifold studied by Candelas, Witten and Vafa, namely

conifold.



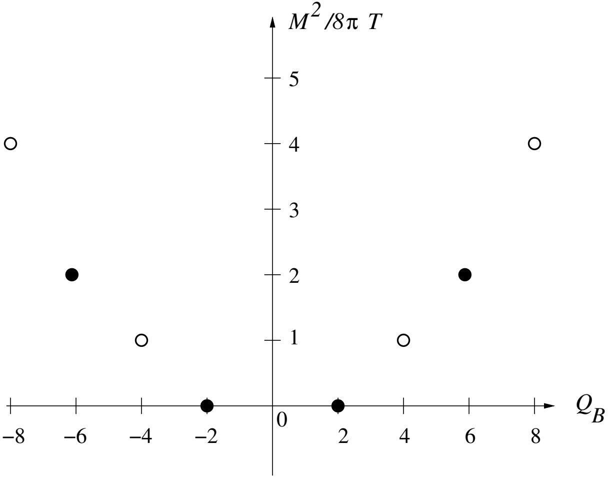
Our goal:

Study states of closed string propagating on

$$R_4 \times Y_6, \qquad Y_6 = \text{conifold}$$

and interpret them as hadrons in 4D $\mathcal{N} = 2$ QCD.

Spectrum of spin-0 and spin-2 states as a function of the baryonic charge. Closed and open circles denote spin-0 and spin-2 states, respectively.



Conclusions

- Worldsheet internal dynamics of non-Abelian string in U(N) gauge theory with $N_f = N$ flavors is described by $CP(N-1)$ model
- Non-Abelian confined monopole = $CP(N-1)$ kink
- 2D-4D correspondence: exact BPS spectrum in quark vacuum of $\mathcal{N} = 2$ 4D Seiberg-Witten theory coincides with BPS spectrum of $\mathcal{N} = (2,2)$ 2D $CP(N-1)$ model
- In quark vacuum we have
  "Instead-of-confinement" phase
  Higgs-screened quarks and gauge bosons evolve into monopole-antimonopole stringy mesons.
- "Instead-of-confinement" phase is rather close to what we observe in the real-world QCD.
- For $N = 2$, $N_f = 4$ non-Abelian vortex behaves as a critical superstring