Off-diagonal functoriality of DeWitt series, expansions for functions of differential operators, and Mellin-Barnes representations

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Heat kernel

Heat kernel (HK) $K_F(\tau|x,x')$ of operator $F(\nabla)$:

$$K_F(\tau|x,x') \equiv e^{-\tau F(\nabla)}\delta(x,x').$$

Extremely useful in QFT/QG, e.g. to calculate effective action:

$$\mathit{\Gamma}^{1-\text{loop}}[\phi] = \frac{1}{2}\log\,\mathrm{Det}\,F(\nabla) = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}\,K_F(\tau|x,x'), \qquad F(\nabla) \sim \frac{\delta^2 S[\phi]}{\delta\phi\,\delta\phi}.$$

DeWitt expansion for HK of Laplace-type operator

$$F(\nabla) = -\Box + N_a(x)\nabla^a + P(x), \qquad \Box \equiv g_{ab}\nabla^a\nabla^b.$$

Asymptotic **DeWitt expansion** at $\tau \to 0\,(\Longleftrightarrow$ UV):

$$e^{-\tau F(\nabla)}\delta(x,x') = \frac{1}{(4\pi\tau)^{d/2}}\exp\left(-\frac{\sigma(x,x')}{2\tau}\right)\sum_{n=0}^{\infty}\tau^n\,a_n(F|x,x'),$$

here:

- $\sigma(x,x') \equiv \frac{1}{2}[\mathrm{dist}(x,x')]^2$ is the Synge world function,
- $a_n(F|x,x')$ are DeWitt coefficients—**contain all background data**.

DeWitt series is not enough: one needs expansions for $f(F)\,\delta(x,x')$

HK for higher-order minimal operators

$$H(\nabla) = F^\nu(\nabla) + W(\nabla), \qquad W(\nabla) = O(\nabla^{2\nu-1}).$$

HK is constructed perturbatively [Barvinsky, Kurov, Wachowski, 2024],

$$e^{-\tau H(\nabla)}\delta(x,x') = U(\tau|\nabla)\, e^{-\tau F^\nu(\nabla)}\delta(x,x'), \qquad U(\tau|\nabla) \sim \sum_n W^n(\nabla).$$

HK for nonminimal operators

HK for nonminimal operators (principal part $\neq \Box^\nu$) has a more complicated structure, e.g. for Proca [Barvinsky, AK, 2024]:

$$e^{\tau(\Box\delta^a_b - \nabla^a\nabla_b - R^a_b)}\delta(x,x') = e^{\tau(\Box\delta^a_b - R^a_b)}\delta(x,x') + \nabla^a\nabla_c\Big[\,\frac{e^{\tau\Box}-1}{\Box}\,\delta(x,x')\Big]^c_b.$$

Kernels $F^{-\mu}e^{-\tau F^\nu}\delta(x,x')$, $\mu,\nu\in\mathbb{Z}_{>0}$ act as building blocks for nonminimal heat kernels [Barvinsky, AK, Wachowski, 2025].

$$f(F)\,\delta(x,x') = ?$$

Main idea: off-diagonal expansion over $a_n(F|x,x')$

All* the data regarding geometry and F is contained inside $a_n(F|x,x')$, hence

$$e^{-\tau\hat{F}}\delta(x,x') = \sum_{n=0}^{\infty} \frac{\tau^{n-d/2}e^{-\frac{\sigma}{2\tau}}}{(4\pi)^{d/2}}\,\hat{a}_n(F|x,x'), \quad f(\hat{F})\delta(x,x') = \sum_{n=0}^{\infty} \mathbb{B}_{d/2-n}[f|\sigma]\,\hat{a}_n(F|x,x').$$

"**Basis kernels**" $\mathbb{B}_m[f|\sigma]$—functionals of f and functions of σ
Same coefficients $a_n(F|x,x')$ for any function $f(F)$—**"Off-diagonal functoriality."**

Expression for $\mathbb{B}_\alpha[f|\sigma]$

$$f(\hat{F}) = \mathfrak{L}_f\, e^{-\tau\hat{F}} \equiv \int_0^\infty d\tau\, \bar{f}(\tau)\, e^{-\tau\hat{F}},$$

basis kernels $\mathbb{B}_\alpha[f|\sigma]$ are obtained by applying integral transforms to DeWitt series term-by-term:

$$\mathbb{B}_\alpha[f|\sigma] = \mathfrak{L}_f\left[\frac{\tau^{-\alpha}}{(4\pi)^{d/2}}\,e^{-\frac{\sigma}{2\tau}}\right] = \int_0^{\infty} d\tau\, \bar{f}(\tau)\frac{\tau^{-\alpha}}{(4\pi)^{d/2}}\,e^{-\frac{\sigma}{2\tau}}; \qquad \alpha = \frac{d}{2} - n.$$

Two main challenges

- $\mathbb{B}_\alpha[f|\sigma]$ are typically complicated special functions of σ
- $\mathbb{B}_\alpha[f|\sigma]$ can contain IR divergences

Mellin-Barnes representation

Both challenges are addressed by using **Mellin-Barnes** (MB) representation for $\mathbb{B}_\alpha[f|\sigma]$.

MB integrals are $\mathbb{C}$-integrals of the type

$$\mathrm{MB}_1(z) = \int_C \frac{ds}{2\pi i} z^{-s} \frac{\prod_i \Gamma(a_i s + b_i)}{\prod_j \Gamma(c_i s + d_i)}.$$

In MB representation IR divergences appear as **contour pinches**, e.g.

$$\mathbb{B}_\alpha\Big[\frac{e^{-\tau F^\mu}}{F^\nu}\Big|\sigma\Big] = \frac{\tau^{\frac{\mu-\alpha}{\nu}}}{(4\pi)^{d/2}} \int \frac{ds}{2\pi i} h(s) \left(\frac{\sigma}{2\tau^{1/\nu}}\right)^{-s}$$

$$h(s) = \frac{\Gamma(s)\Gamma\left(\frac{\alpha-s-\mu}{\nu}\right)}{\nu\Gamma(\alpha-s)}$$

—pinch between red and blue poles.

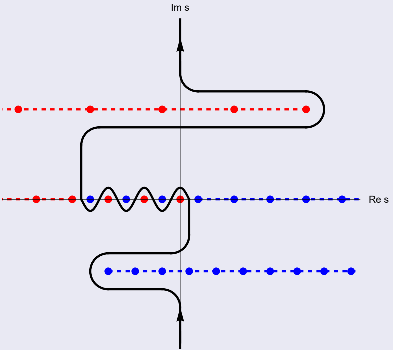

Figure: MB contour for $\Gamma(s)\Gamma(2s+2)\Gamma(-s-5/2)\Gamma(-s-2)$

Regulating IR divergences

Pinch regularization

$$\mathbb{B}_\alpha[f|\sigma] \mapsto \mathbb{B}_{\alpha+\epsilon}[f|\sigma] \approx \text{dim reg}$$

Massive regularization

$$\mathbb{B}_\alpha[f(F)|\sigma] \mapsto \mathbb{B}_\alpha[f(F+m^2)|\sigma]$$

They are consistent

Summary

- A universal method for building **off-diagonal** expansions for $f(F)\,\delta(x,x')$ is suggested,

$$f(F)\,\delta(x,x') = \sum_{n=0}^{\infty} \mathbb{B}_{d/2-n}\big[f\big|\sigma\big]\,a_n(F|x,x')$$

- Basis kernels $\mathbb{B}_\alpha[f|\sigma]$ for various f are obtained and expressed in terms of multiple Mellin-Barnes integrals
- Depending on f, $\mathbb{B}_\alpha[f|\sigma]$ may contain IR divergences = MB contour pinches
- Consistency of massive and pinch regularizations in this procedure is shown

Publications

- Barvinsky, Kalugin, Wachowski, *Phys. Lett. B*, 874:140291, 2026.
- Barvinsky, Kalugin, Wachowski, *Phys. Rev. D*, 113:045005, 2026.
- Barvinsky, Kalugin, Wachowski, Submitted to *Phys. Rev. D*.

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