On the Dirac-Bergmann procedure for Hamiltonian formalism of gauge systems

Kirill Russkov[1,2]

[1]Saint Petersburg State University

[2]Petersburg Nuclear Physics Institute

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Motivation

- While the Dirac-Bergmann algorithm is a procedure originally developed for quantization of constrained systems, it is also widely used for non-perturbative counting of degrees of freedom.
- In the second case, however, there are several subtleties that rarely receive due attention in the literature, such as the use of the Dirac conjecture and the extended Hamiltonian.
- This work aims to provide a pedagogical exposition of the Dirac-Bergmann algorithm for Hamiltonian analysis of gauge theories drawing attention to the aforementioned nuances.

What is a constraint?

In the Lagrangian formalism, a constraint is a Lagrange equation that involves only the generalized coordinates and generalized velocities, and contains no second time derivatives of the coordinates:

$$f(q, \dot{q}) = 0. \tag{1}$$

In the Hamiltonian formalism, a constraint is a condition of the form

$$f(q, p) = 0. \tag{2}$$

Such a condition restricts the possible choice of initial data $(q(t_0), \dot{q}(t_0))$ in the Lagrangian framework or $(q(t_0), p(t_0))$ in the Hamiltonian.

Electrodynamics as an example of a constrained system

Every gauge system is, in fact, a system.

1.

$$\partial_\mu F^{\mu\nu} = 0 \iff \begin{cases} \partial_0 F_{0i} - \partial_j F_{ji} = 0, \\ \partial_i F_{0i} = 0. \end{cases} \tag{3}$$

- $\nu = i$ are second-order equations
- $\nu = 0$ is a constraint, since it contains no second time derivatives:

$$\partial_i F_{0i} = \partial_i(\partial_0 A_i - \partial_i A_0) = 0. \tag{4}$$

2. In GR, the vacuum Einstein equations $G^{0\mu} = 0$ are constraints since they do not contain $\partial_0^2 g_{\mu\nu}$:

$$G^{0\mu} = \frac{1}{2}\varepsilon^{0\rho\sigma\lambda}\varepsilon^{\mu\alpha\beta}{}_{\lambda} R_{\rho\sigma\alpha\beta} = 0 \tag{5}$$

Reminder of the Dirac–Bergmann algorithm

Gauge systems have a degenerate Hessian of the Lagrangian with respect to the velocities:

$$rank\ \frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j} = r < n. \tag{6}$$

This makes the transition to the Hamiltonian formalism nontrivial, since it is impossible to express all velocities in terms of the canonical momenta using

$$p^i = \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}). \tag{7}$$

Therefore, the definitions (7) give rise to constraints on the coordinates and momenta:

$$\phi^a(q, p) = p^a - f^a(q, p^A) = 0, \quad a = 1 \ldots n - r. \tag{8}$$

These constraints are called *primary* constraints.

Reminder of the Dirac–Bergmann algorithm

In order to obtain the Hamiltonian EOMs equivalent to the Lagrange equations one has to construct the *total Hamiltonian*:

$$H_T(q, p, \lambda) = H_{can}(q, p^B) + \lambda_a \phi^a(q, p). \tag{9}$$

The Hamiltonian equations of motion are

$$\dot{q}_i = \{q_i, H_T\}, \qquad \dot{p}^i = \{p^i, H_T\}, \qquad \phi^a(q, p) = 0. \tag{10}$$

For the primary constraints to hold at all times, $\phi^a(q(t), p(t)) = 0$, we must require

$$\dot{\phi}^a = \{\phi^a, H_T\} = 0. \tag{11}$$

This condition

- may be satisfied identically
- or impose restrictions on the Lagrange multipliers
- or give rise to new constraints $\chi^u(q, p) = 0$, which are called *secondary constraints*.

Classification of constraints

Dirac suggested a useful classification of constraints:

- If $\{\chi_1, \chi_2\} = 0$, then χ_1 and χ_2 are *first-class* constraints.
- If $\{\chi_1, \chi_2\} \neq 0$, then χ_1 and χ_2 are *second-class* constraints.

Reminder of Dirac's recipe

1. Dirac's conjecture is the statement that all first-class constraints are independent generators of gauge transformations:

$$\delta g = \varepsilon^1_a\{g, \phi^a_1\} + \varepsilon^2_a\{g, \phi^a_2\}, \quad \varepsilon^{1,2}_a \; = \text{independent gauge parameters.} \tag{12}$$

$$\phi^a_1 = \text{primary 1st class}, \qquad \phi^a_2 = \text{secondary 1st class} \tag{13}$$

2. Dirac suggested transitioning from the total Hamiltonian to the *extended Hamiltonian*:

$$H_E = H_T + \lambda^2_a\phi^a_2. \tag{14}$$

It is claimed that, after this transition, the system remains equivalent to the original one.

Counting degrees of freedom

- *Total number of modes* N_{tot}: the total number of variables in the Lagrangian formulation.
- *Number of gauge modes* N_{gauge}: the number of independent gauge parameters.
- *Number of dynamical modes* N_{dyn}: the minimal number of independent Cauchy initial data for non-pure-gauge variables, divided by two.
- *Number of constrained modes* $N_{con} \equiv N_{tot} - N_{gauge} - N_{dyn}$

$$N_{tot} = N_{dyn} + N_{gauge} + N_{con}. \tag{15}$$

- For simplicity, we consider only the class of theories with $N_{con} = N_{gauge}$, the so-called "gauge hits twice" case

Why constrained modes matter

Constrained modes correspond to non-dynamical gauge-invariant quantities. Examples:

- Longitudinal component of electric field $E_i = E_i^L + E_i^T$:

  E_i^L is a constrained mode that describes Coulomb's law
- 2 scalar modes and 2 vector modes in cosmological perturbation theory:

$$ds^2 = a^2(\eta)\left[-(1+2\Phi)\,d\eta^2 + 2S_i^T\,d\eta\,dx^i + \left((1-2\Psi)\delta_{ij} + h_{ij}^{\mathrm{TT}}\right)dx^i dx^j\right] \tag{16}$$

$$\phi, \psi, S_i^T = 4 \text{ constrained modes} \tag{17}$$

The Dirac conjecture

- The Dirac conjecture generally does not hold for the total Hamiltonian. The correct generator is given by a specific combination of both primary and secondary first-class constraints:

$$G_T = \dot{\varepsilon}_a \phi_1^a + \varepsilon_a F^a[\phi_1, \phi_2], \tag{18}$$

  where ε is a gauge parameter, $F[\phi_1, \phi_2]$ is a certain combination of primary and secondary first-class constraints.

- However, the Dirac conjecture is valid for the extended Hamiltonian

$$G_E = \varepsilon_a \phi_1^a + \zeta_a \phi_2^a. \tag{19}$$

The extended Hamiltonian

$$H_E = H_T + \lambda_a^2 \phi_2^a$$

- Naively, the extended Hamiltonian formulation modifies the system's EOM in a way that turns N_{con} into N_{gauge}
- Example: In ED, Gauss's law changes to $\partial_i E_i = 0 \to \partial_i E_i = -\vec{\partial}^2 \lambda_2$.

  Now the longitudinal component becomes unphysical.
- Therefore, we obtain $2N_{gauge}$ modes instead of N_{gauge}

The extended Hamiltonian

The correct approach consists of a Stueckelberg-like procedure:

$$q_i = q_i^E - \lambda_a^2 \{q_i^E, \phi_1^a\}, \qquad p^i = p_E^i - \lambda_a^2 \{p_E^i, \phi_1^a\}, \tag{20}$$

We artificially increase the number of variables the number of variables to introduce an additional symmetry.

This is equivalent to a redefinition of gauge invariants

$$F = gauge - invariant \iff \{F, \phi_{1,2}^a\} = 0 \quad \text{on-shell} \tag{21}$$

Constrained modes are now encoded in the new gauge-invariants.

Conclusion

We have shown that

1. Dirac's conjecture does not hold for the total Hamiltonian, but does hold for the extended Hamiltonian[1].
2. In general, the extended Hamiltonian the dynamics of the original Lagrangian variables, turning constrained modes into gauge modes.
3. In order for the dynamics of invariant quantities to remain unchanged, one has to redefine the variables in a specific way. This restores the equivalence with the Lagrangian system.
4. This redefinition is equivalent to a change of variables in the first-order action, resembling the Stueckelberg procedure.
5. We illustrate the discussion with examples from Yang–Mills theories and GR.

[1] Dirac's conjecture holds in both cases if gauge transformations are considered only at a fixed instant in time.

Stueckelberg Procedure

Consider a massive vector field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{m^2}{2}A_\mu A^\mu. \tag{22}$$

The equations of motion are

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \tag{23}$$

Unlike electrodynamics, this model does not possess a $U(1)$ gauge symmetry.

Nevertheless, one can artificially restore gauge symmetry by introducing the reparameterization

$$A_\mu = ie\ \tilde{A}_\mu +\ \partial_\mu \phi. \tag{24}$$

The transformations

$$\tilde{A}_\mu \to \tilde{A}_\mu - \partial_\mu \alpha, \quad \phi \to e^{ie\alpha}\phi \tag{25}$$

are then symmetries of the system.

Generators of Local Symmetries

Anderson–Bergmann–Castellani theorem. (L. Castellani, *Symmetries in constrained Hamiltonian systems*, Annals of Physics, 143:357, 1982.)
Suppose there exists an infinitesimal transformation of the canonical variables,

$$q \to q + \delta q, \quad p \to p + \delta p,$$

such that both the original variables (q, p) and the transformed variables $(q + \delta q, p + \delta p)$ satisfy the constrained Hamilton equations.
Then the generator of this transformation has the form

$$G = \sum_{k=0}^{m} \varepsilon^{(m-k)}(t)\, G_k, \tag{26}$$

where the G_k are first-class quantities satisfying the following chain of relations:

Castellani Generators

$$
\begin{aligned}
G_0 &= \text{LC of primary constraints}, \\
G_0 + \{G_1, H_T\} &= \text{LC of primary constraints}, \\
G_1 + \{G_2, H_T\} &= \text{LC of primary constraints}, \\
&\vdots \\
G_{m-1} + \{G_m, H_T\} &= \text{LC of primary constraints}, \\
\{G_m, H_T\} &= \text{LC of primary constraints}.
\end{aligned} \tag{27}
$$

Castellani showed that the "basic generators"[2] correspond to chains of minimal length. This minimality is achieved by redefining some of the quantities G_k through the addition of a linear combination of primary constraints:

$$G_k \to G_k + \alpha_a \phi^a.$$

We will refer to the generators obtained in this way as Castellani generators G_C.

[2]That is, those which cannot be represented as a nontrivial linear combination of other generators.