Basics of stellar evolution theory I

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This lecture is based on the following materials.

An Introduction to the Theory of Stellar Structure and Evolution (D. Prialnik, Cambridge University Press, 2009)

Stellar Structure and Evolution (R. Kippenhahn, A. Weigert, & A. Weiss, Springer, 2012)

https://www.astro.ru.nl/~onnop/education/stev_utrecht_notes/

Stellar evolution

- What is a star?
    - an object bound by self-gravity
    - an object radiating energy supplied by an internal source

Sun?

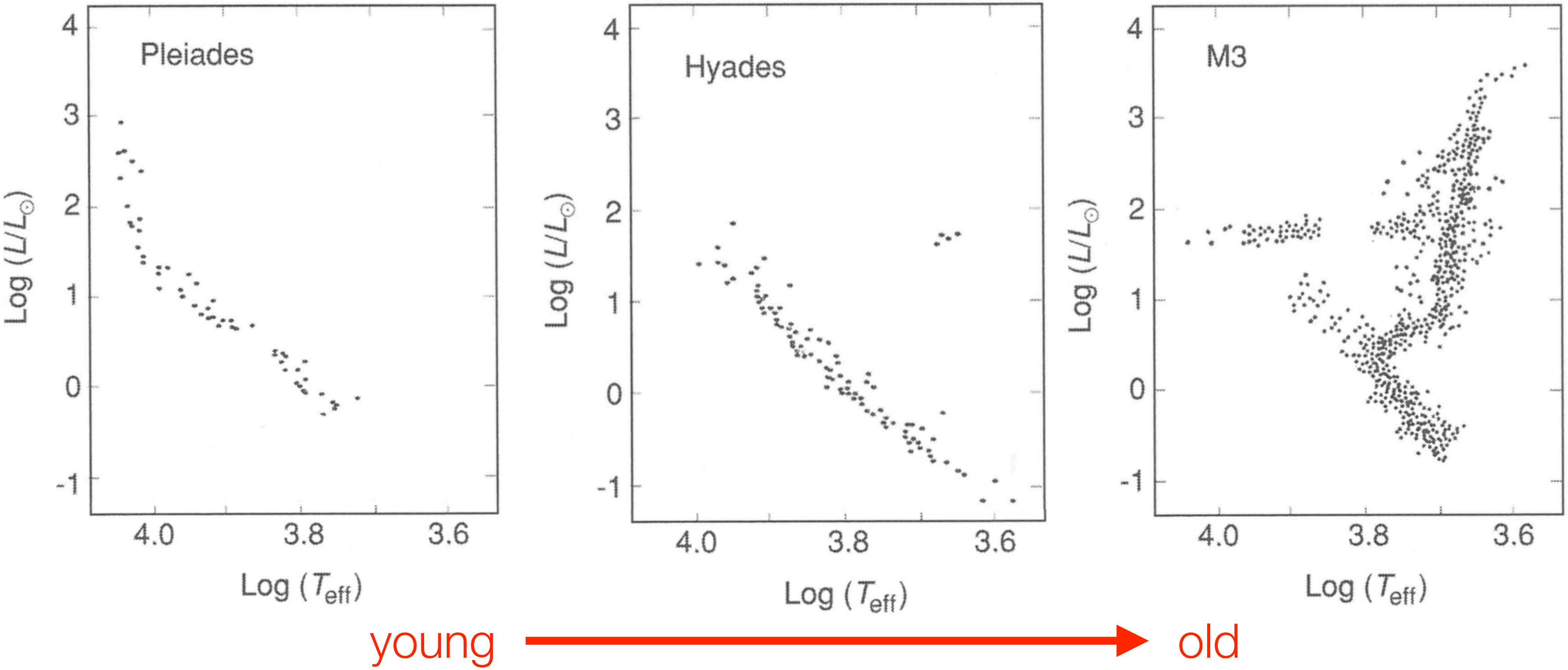
Earth?

comets?

supernovae?

Stellar evolution

- What is evolution?
  - stars keep releasing their internal energy
  - some changes in stars are unavoidable!



Basic equations

Assumptions in this lecture
1. spherical symmetry
2. hydrostatic

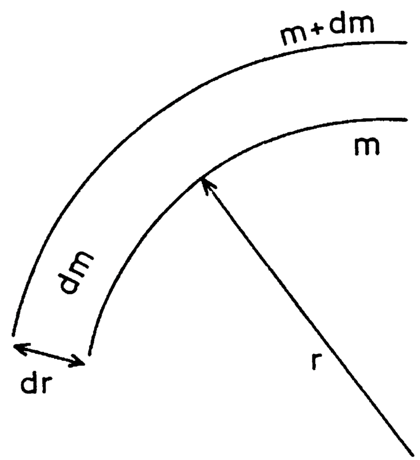
Conservation of mass

$$dm(r) = \rho(r)dV(r) = \rho(r)4\pi r^2 dr$$

- integrating from the center to a radius *r*, one gets

$$m(r) = \int_0^r \rho 4\pi r^2 dr$$

- *m(r)* increases monotonically with *r*, we can use *m(r)* (or simply *m*) as a radial coordinate instead of *r*.



Equation of motion

$$\Delta m \frac{\partial^2 r}{\partial t^2} = F_g + P(r)dS - P(r+dr)dS$$

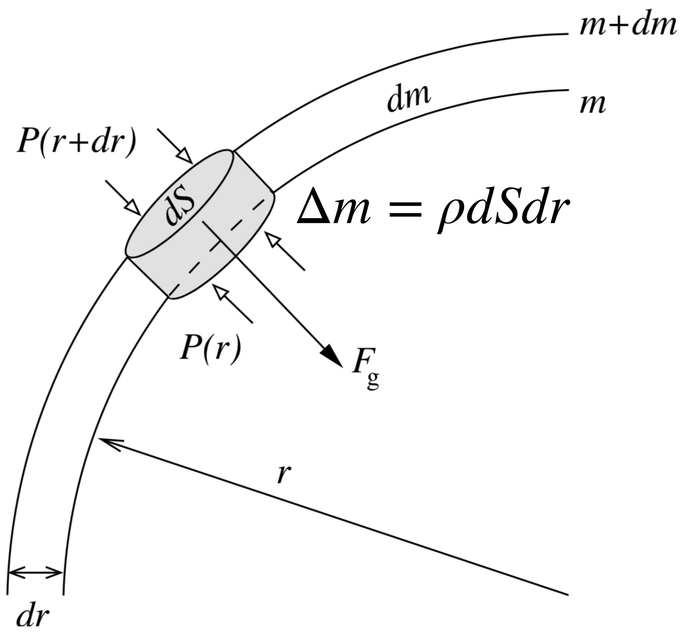
$$F_g = -\frac{Gm\Delta m}{r^2}$$

$$P(r+dr) \simeq P(r) + \frac{\partial P}{\partial r}dr$$

$$P(r)dS - P(r+dr)dS = -\frac{\partial P}{\partial r}drdS$$

$$\Delta m \frac{\partial^2 r}{\partial t^2} = -\frac{Gm\Delta m}{r^2} - \frac{\Delta m}{\rho}\frac{\partial P}{\partial r}$$

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}$$

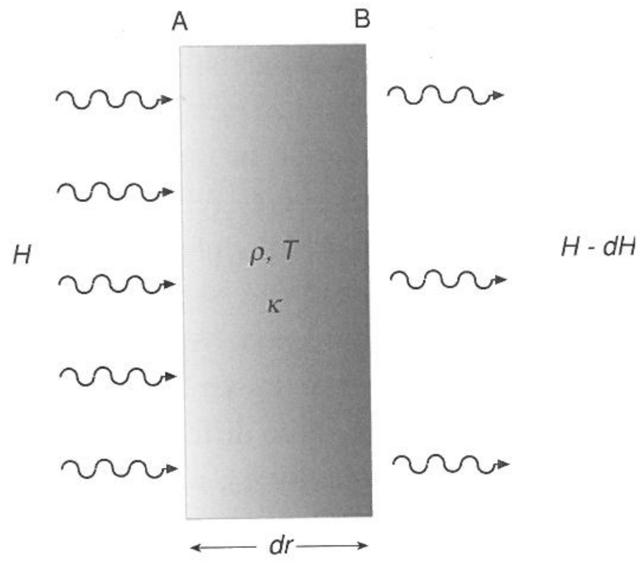


$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

$$dm = 4\pi r^2 dr$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

Radiation transfer



H : radiation flux (energy per unit area per unit time)

$$dH = -\kappa H \rho dr$$

κ : opacity

momentum gained by absorbing radiation: $\dfrac{|dH|}{c}$

Radiation transfer

$$P_{\rm rad}(r) - P_{\rm rad}(r+dr) = \frac{|dH|}{c}$$

$$P_{\rm rad}(r) - P_{\rm rad}(r+dr) = -\frac{dP_{\rm rad}}{dr}dr \qquad dH = -\kappa H \rho dr$$

$$\frac{H\kappa\rho}{c} = -\frac{dP_{\rm rad}}{dr}$$

$$P_{\rm rad} = \frac{1}{3}aT^4$$

$$H = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

Radiation transfer

$$F = 4\pi r^2 H = -4\pi r^2 \frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

F : total flux crossing a spherical surface

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{F}{4\pi r^2}$$

$$\boxed{\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2}}$$

Basic equations: summary

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$: mass conservation

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$: equation of motion

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2}$$: radiation transport

$$\frac{dF}{dm} = q$$: heat generation (e.g., nuclear reactions)

Timescales

ϕ : a physical parameter

$\dot{\phi}$: rate of change of ϕ

$$\tau \simeq \frac{\phi}{\dot{\phi}}$$

timescale for the physical process

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2}$$

$$\frac{dF}{dm} = q$$

Dynamical timescale

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \qquad \phi = R \qquad \dot{\phi} \simeq v_{\rm esc} = \sqrt{\frac{2GM}{R}}$$

$$\tau_{\rm dyn} \sim \frac{R}{v_{\rm esc}} = \sqrt{\frac{R^3}{2GM}}$$

$$\tau_{\rm dyn} \sim 10^3 \left(\frac{R}{R_\odot}\right)^{3/2} \left(\frac{M}{M_\odot}\right)^{-1/2} \ \sec$$

Thermal timescale (Kelvin-Helmholtz timescale)

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2} \qquad \phi = U \sim \frac{GM^2}{R} \qquad \dot{\phi} = L$$

$$\tau_{\rm th} = \frac{U}{L} \sim \frac{GM^2}{RL}$$

$$\tau_{\rm th} \sim 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1} \ {\rm sec}$$

$$\sim 3 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1} \ {\rm yr}$$

Nuclear timescale

$$\frac{dF}{dm} = q \qquad \phi = \varepsilon M c^2 \qquad \dot{\phi} = L$$

$$\tau_{\rm nuc} = \frac{\varepsilon M c^2}{L}$$

$$\tau_{\rm nuc} \sim 5 \times 10^{17} \left(\frac{\varepsilon}{10^{-3}}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1} \ \text{sec}$$

$$\sim 10^{10} \left(\frac{\varepsilon}{10^{-3}}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1} \ \text{yr}$$

Timescales

$$\tau_{\rm dyn} \sim 10^3 \left(\frac{R}{R_\odot}\right)^{3/2} \left(\frac{M}{M_\odot}\right)^{-1/2} \ \text{sec}$$

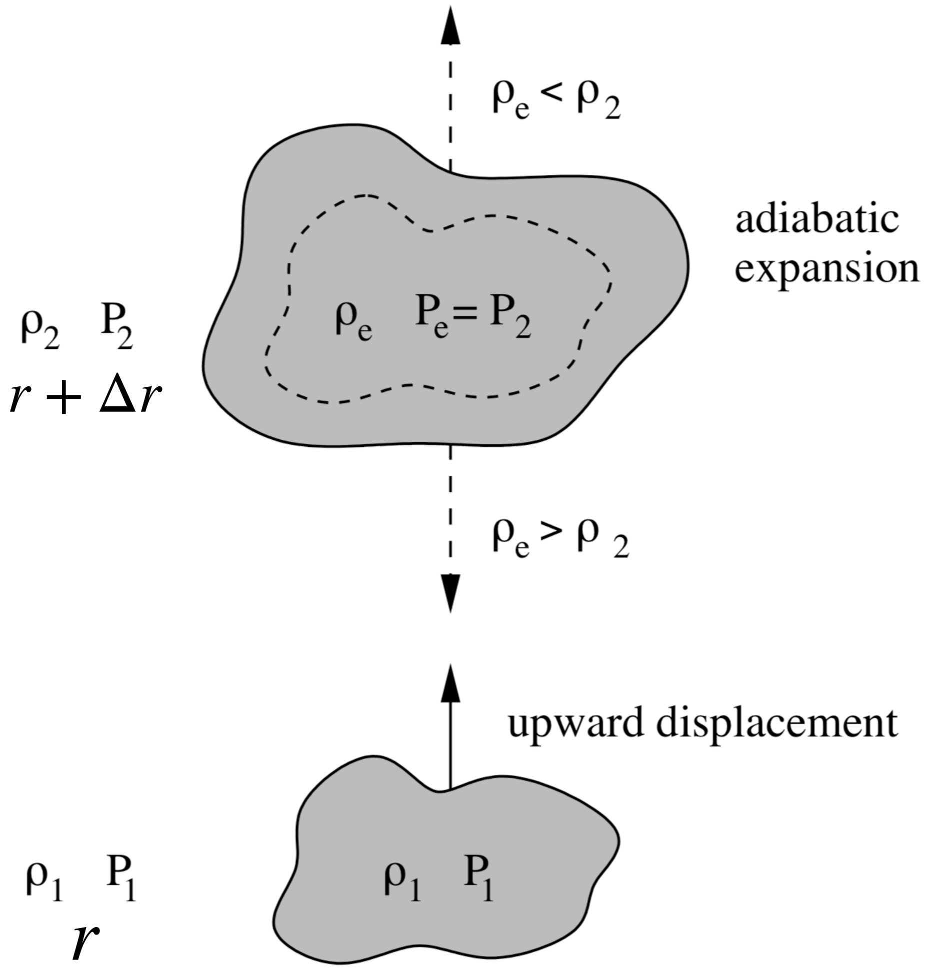
$$\tau_{\rm th} \sim 3 \times 10^7 \left(\frac{M}{M_\odot}\right)^{2} \left(\frac{R}{R_\odot}\right)^{-1} \left(\frac{L}{L_\odot}\right)^{-1} \ \text{yr}$$

$$\tau_{\rm nuc} \sim 10^{10} \left(\frac{\varepsilon}{10^{-3}}\right) \left(\frac{M}{M_\odot}\right) \left(\frac{L}{L_\odot}\right)^{-1} \ \text{yr}$$

$$\tau_{\rm dyn} \ll \tau_{\rm th} \ll \tau_{\rm nuc}$$

Convection

- energy transport by matter mixing, not by radiation



$$\frac{\delta P_e}{P_e} = \gamma_a \frac{\delta \rho_e}{\rho_e}$$

$$\delta P_e = \frac{dP}{dr}\Delta r \qquad \delta \rho_e = \frac{\rho_e}{P_e}\frac{1}{\gamma_a}\frac{dP}{dr}\Delta r$$

$$\rho_e = \rho_1 + \delta \rho_e \qquad \rho_2 = \rho_1 + \frac{d\rho}{dr}\Delta r$$

$\rho_e < \rho_2$: unstable

$$\frac{d \ln \rho}{d \ln P} < \frac{1}{\gamma_a}$$

Equation of state

gas pressure (ions, electrons)
electron degenerate pressure (non-relativistic, relativistic)
radiation pressure

Ion gas pressure

$$P = \frac{1}{3}\int_0^\infty v_p p n(p) dp$$

Maxwellian velocity distribution

$$n(p)dp = \frac{n_{\text{ion}}}{(2\pi m_{\text{ion}} k_B T)^{3/2}} e^{-\frac{p^2}{2m_{\text{ion}} k_B T}} 4\pi p^2 dp$$

$$P_{\text{ion}} = n_{\text{ion}} k_B T$$

Ion gas pressure

$$P_{\rm ion} = n_{\rm ion} k_B T$$

$$n_{\rm ion} = \sum_i n_i = \sum_i \frac{X_i \rho}{A_i m_p}$$

X_i : mass fraction of *i*th element A_i : mass number of *i*th element

$$\frac{1}{\mu_{\rm ion}} \equiv \sum_i \frac{X_i}{A_i} \qquad \mu_{{\rm ion},\odot} \simeq 1.3$$

$$P_{\rm ion} = n_{\rm ion} k_B T = \frac{k_B}{m_p} \frac{1}{\mu_{\rm ion}} \rho T = \frac{\mathfrak{R}}{\mu_{\rm ion}} \rho T$$

$$\mathfrak{R} \equiv \frac{k_B}{m_p}$$:ideal gas constant

Electron gas pressure

$$P_{\rm e} = n_{\rm e} k_B T$$

$$n_{\rm e} = \sum_i Z_i n_i = \sum_i Z_i \frac{X_i \rho}{A_i m_p} = \frac{\rho}{\mu_e m_p}$$

$$\frac{1}{\mu_{\rm e}} \equiv \sum_i X_i \frac{Z_i}{A_i}$$

μ_e :average number of free electrons per nucleon

$$\mu_{{\rm e},\odot} \simeq 1.2$$

$$P_{\rm e} = \frac{\Re}{\mu_e} \rho T$$

Total gas pressure

$$P_{\text{gas}} = P_{\text{ion}} + P_{\text{e}}$$

$$= \left(\frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_{\text{e}}} \right) \mathfrak{R} \rho T$$

$$= \frac{\mathfrak{R}}{\mu} \rho T$$

$$\frac{1}{\mu} \equiv \frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_{\text{e}}}$$

$$P_{\text{gas}} = \frac{\mathfrak{R}}{\mu} \rho T$$

Electron degeneracy pressure

- Heisenberg uncertainty principle

$$\Delta V \Delta^3 p \geq h^3$$

Electron degeneracy pressure

- Heisenberg uncertainty principle

$$\Delta V \Delta^3 p \geq h^3$$

- Pauli exclusion principle
    - no electrons can occupy the same quantum state

$$n_{\rm e}(p) dp dV = 2 \frac{4\pi p^2 dp dV}{h^3}$$

$$n_{\rm e}(p) dp = \frac{8\pi p^2}{h^3} dp$$

Electron degeneracy pressure

- completely degenerate electron gas

$$n_e(p) = \frac{8\pi p^2}{h^3} \quad (p \le p_F)$$

$$= 0 \qquad (p > p_F)$$

$$n_e = \int_0^{p_F} n_e(p)dp$$

$$p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$$

Electron degeneracy pressure

- completely degenerate electron gas

$$P = \frac{1}{3}\int_0^\infty v_p p n(p) dp$$

$$n_\mathrm{e}(p)dp = \frac{n_\mathrm{e}}{(2\pi m_\mathrm{e} k_B T)^{3/2}} e^{-\frac{p^2}{2m_\mathrm{e} k_B T}} 4\pi p^2 dp$$

$$\downarrow$$

$$n_\mathrm{e}(p)dp = \frac{8\pi p^2}{h^3} dp$$

Electron degeneracy pressure

- completely degenerate electron gas

$$P = \frac{1}{3}\int_0^\infty v_p p n(p) dp$$

$$P_{\mathrm{e,deg}} = \frac{8\pi p_F^5}{15 m_\mathrm{e} h^3} = \frac{3^{2/3} h^2}{20\pi^{2/3} m_\mathrm{e} m_p^{5/3}} \mu_\mathrm{e}^{-5/3} \rho^{5/3}$$

$$P_{\mathrm{e,deg}} = K_1 \rho^{5/3}$$

Electron degeneracy pressure

- relativistic case

$$P = \frac{1}{3}\int_0^\infty v_p p n(p) dp$$

$$n_e(p) = \frac{8\pi p^2}{h^3} \quad (p \le p_F)$$
$$= 0 \quad (p > p_F)$$

$$p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$$

$$p = \frac{m_e v}{\sqrt{1 - v_p^2/c^2}}$$

Electron degeneracy pressure

- relativistic case

$$P = \frac{1}{3}\int_0^\infty v_p p n(p) dp$$

$$v_p \rightarrow c$$

$$P_{\mathrm{e,r-deg}} = \frac{3^{1/3}hc}{8\pi^{1/3}m_p^{4/3}}\mu_{\mathrm{e}}^{-4/3}\rho^{4/3}$$

$$P_{\mathrm{e,r-deg}} = K_2\rho^{4/3}$$

Radiation pressure

$$P = \frac{1}{3}\int_0^{\infty} cpn(p)dp$$

$$P_{\rm rad} = \frac{1}{3}\int_0^{\infty} c\frac{h\nu}{c}n(\nu)d\nu$$

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\frac{1}{e^{\frac{h\nu}{k_BT}} - 1}d\nu$$

$$P_{\rm rad} = \frac{1}{3}aT^4 \qquad a = \frac{8\pi^5 k_B^4}{15c^3h^3}$$

Equation of state: summary

$$P_{\text{gas}} = \frac{\mathfrak{R}}{\mu}\rho T$$

$$P_{\text{e,deg}} = K_1 \rho^{5/3}$$

$$P_{\text{e,r-deg}} = K_2 \rho^{4/3}$$

$$P_{\text{rad}} = \frac{1}{3} a T^4$$

Mass-radius relation for degenerate stars

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

$$\frac{0 - P_\mathrm{c}}{R - 0} \sim -\frac{G(M/2)}{(R/2)^2}\bar{\rho} \qquad \bar{\rho} = M/(4\pi R^3/3)$$

$$P_c \sim \frac{3}{2\pi}\frac{GM^2}{R^4}$$

$$P_\mathrm{e,deg} = K_1 \rho^{5/3}$$

$$M \propto R^{-3} \qquad \bar{\rho} \propto M^2$$

Chandrasekhar mass limit

$$P_c \sim \frac{3}{2\pi}\frac{GM^2}{R^4} \qquad P_{\rm e,r-deg} = K_2\rho^{4/3} = K_2'\mu_e^{-4/3}\rho^{4/3}$$

$$\bar{\rho} = M/(4\pi R^3/3)$$

$$M_{\rm r-deg} \sim \frac{3}{128\pi}\left(\frac{hc}{Gm_p^{4/3}}\right)^{3/2}\mu_e^{-2} = 0.2\mu_e^{-2}\ M_\odot$$

$$M_{\rm Ch} = 5.836\mu_e^{-2}\ M_\odot$$

$$\mu_e = 2 \qquad M_{\rm Ch} = 1.46\ M_\odot$$

Summary

- Basic equations

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \quad \frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \quad \frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{F}{(4\pi r^2)^2} \quad \frac{dF}{dm} = q$$

  - Eddington luminosity, convection
  - timescales

$$\tau_{\rm dyn} \ll \tau_{\rm th} \ll \tau_{\rm nuc}$$

- Equation of states

$$P_{\rm gas} = \frac{\mathfrak{R}}{\mu}\rho T \quad P_{\rm e,deg} = K_1 \rho^{5/3} \quad P_{\rm e,r-deg} = K_2 \rho^{4/3} \quad P_{\rm rad} = \frac{1}{3} a T^4$$

  - Chandrasekhar mass limit $M_{\rm Ch} = 5.836 \mu_e^{-2}\ M_\odot$

Reference

- An introduction to the theory of stellar structure and evolution
  - Prialnik

- Stellar structure and evolution
  - Kippenhahn, Weigert, & Weiss

- Physics, formation and evolution of rotating stars
  - Maeder